1 Finding Trigonometric Derivatives

1.1 The Derivative as a Function

The definition of the derivative as a function is,

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \]

provided the limit exists.\(^2\) Okay, I’m being repetitious here, but it is nonetheless necessary to be reminded of this definition before proceeding forward.

1.2 The Derivative of Sine

I’d like to show that the derivative of the sine function is the cosine function. Showing that this is the case is in fact difficult. But as I always say, these things have already been done and are the result of someone else’s hard work. Seeing what others have done is important, and will hopefully motivate you to do the same.

The explanation and proof outlined here, although not detailed, should be sufficient to get you thinking. However, please take a look at the book’s proof too.

**Work:** First, draw one cycle of the sine function, and it’s derivative\(^3\) on the same graph.

![Graph of sine function and its derivative](image)

Figure 1: The sine function (solid) and its derivative (dashed).

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\(^1\)This document was prepared by Ron Bannon using \LaTeX.\(^2\)Now of course our new function will have a domain that may differ from the originating function. Yes, the derivative may not be defined at all points along the function.

\(^3\)Same method that we’re doing in class. That is, try to find the slopes at some points and then connect the dots.
You should observe that the dashed curve looks like the cosine curve, but this is certainly not a proof, but at least it’s giving us a hint.

To find the derivative of the sine function we will need to use the definition of derivative.

\[
\frac{d}{dx} (\sin x) = \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h}
\]

This limit does not look easy, and you may wonder why its being rewritten in this form.\(^4\)

\[
\frac{d}{dx} (\sin x) = \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h}
= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}
= \lim_{h \to 0} \left[ \frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right]
= \lim_{h \to 0} \left[ \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right]
\]

Still not much progress, but at least I see a simple limit within the bigger problem that we’ve seen before.

\[
\lim_{h \to 0} \frac{\sin h}{h}
\]

If you can recall, you were asked\(^5\) to do this problem numerically and found that it equaled one. Now we need to show this. Here’s one way to do so. This is a geometric argument, but differs from the book’s geometric approach.

\[\text{Figure 2: Will be discussed and labeled in class.}\]

\(^4\)The main reason is that this is what you did in MTH-120, and sum identities were extensively used.

\(^5\)If you’re reading the book and doing the suggested review problems, you’ve seen this limit!
We will label this diagram in class using $a, b, c$ and $\theta \in (0, \pi/2)$, and discuss why $a < b$. We will get the following relationship as well, simply by using trigonometry.

$$\sin \theta < \theta \quad \Rightarrow \quad \frac{\sin \theta}{\theta} < 1$$

Now finding $c$, which simple turns out to be $\tan \theta$ and using an existing theorem that states that $\theta < \tan \theta$ for $0 < \theta < \pi/2$. So now we finally have (this will also be discussed in class).

$$\sin \theta < \theta < \tan \theta \quad \text{for } 0 < \theta < \pi/2$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Now, using the Squeeze Theorem, we will find$^7$ that.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

That’s a lot of work. But as you can see, we also need.

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$$

Which will easily be shown, in class, to be equal to zero. Finally, we have what we expected from the beginning.

$$\frac{d}{dx} (\sin x) = \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h \to 0} \left[ \frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[ \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right]$$

$$= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$= \cos x$$

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$^6$This proof is not easy, see page A43 of your textbook if interested.

$^7$We really just showed that this is true from the right of zero, but fortunately this is an even function, so we’ve really showed it from both the right and left.
1.3 Insightful Graphs

Some graphs, although not proofs, over insight.

![Figure 3: This is $y = \tan x$ and $y = x$.](image)

Certainly, when one looks at this graph it may not be so clear that $\tan x > x$ for $0 < x < \pi/2$. This is why we need to prove things.

![Figure 4: This is $y = \frac{\sin x}{x}$.](image)

Now, I really think this graph is convincing, but we still need to prove it as we did today.
2 The Remaining Rules and Examples

Show all work and box the final answer.

1. You *should* be able to prove using the definition of the derivative that
\[
\frac{d}{dx} (\cos x) = -\sin x,
\]

2. Using rules you should be able to show that
\[
\frac{d}{dx} (\tan x) = \sec^2 x.
\]

3. Using rules you should be able to show that
\[
\frac{d}{dx} (\csc x) = -\cot x \csc x.
\]
4. Using rules you should be able to show that
\[
\frac{d}{dx} (\sec x) = \tan x \sec x.
\]

5. Using rules you should be able to show that
\[
\frac{d}{dx} (\cot x) = -\csc^2 x.
\]

6. Find an equation of the tangent line to the curve \( y = \sec x - 2 \cos x \) at the point \( (\pi/3, 1) \).

7. Differentiate.
\[
y = \frac{x + \sin x}{x^2 + \cos x}
\]
8. Find the limit.
\[
\lim_{x \to 0} \frac{\sin^2 3x}{x^2}
\]

9. Find the limit.
\[
\lim_{x \to 0} \frac{1 - \tan x}{\sin x - \cos x}
\]

10. Suppose \( f(\pi/3) = 4 \) and \( f'(\pi/3) = -2 \), and let \( g(x) = f(x) \sin x \) and \( h(x) = \cos x \div f(x) \). Find \( g'(\pi/3) \) and \( h'(\pi/3) \).

11. If \( f(\beta) = \beta \cdot \sin \beta \), find \( f'(\beta) \) and \( f''(\beta) \).