1 l’Hôpital’s Rule

Guillaume de l’Hôpital, a French mathematician, published (1696) the first successful differential calculus book, *Analyse des Infiniment Petits*, which I believe literally translates to: Analysis of the Infinitely Small. However, he is better known for a rule that bears his name, but the rule was actually the work of a Swiss mathematician, named Johann Bernoulli. Let’s take a look . . .

1.1 An Example Limit

Suppose you are asked to compute the following limit.

\[
\lim_{x \to 0} \frac{e^{2x} - 1}{x}
\]

Substituting in \( x = 0 \) gives \( 0/0 \), which is undefined and cancelations are not algebraically obvious as there were before. Let’s instead look at the graph and try to make a guess. Clearly we know that the function is not defined at zero, but to the left and right of zero it appears (see graph) that the function is going towards 2 as \( x \) gets close to 0. You may recall a similar trigonometric (again cancelations are not algebraically obvious) limit that that you’ve seen before.

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

It was a rather long and lengthy geometric argument that showed that limit exists and is 1. Geometric arguments are nice, and l’Hôpital’s rule may best be understood by looking at the geometry, but in a rather different way.

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\( ^1 \)This document was prepared by Ron Bannon using \LaTeX\ 2ε.
In a geometric sense we are going to return to using derivatives to create local lines that fit the numerator and denominator at the point of interest. For example, the numerator $e^{2x} - 1$ looks very much like

$$y = 2x$$

when $x = 0$, and of course the denominator $x$, always looks like a line. So when we’re near zero, the numerator and denominator ratio becomes approximately the same as

$$\frac{e^{2x} - 1}{x} \approx \frac{2x}{x} = \frac{2}{1}.$$  

You should note that if we let

$$f(x) = e^{2x} - 1,$$

and

$$g(x) = x,$$

we have

$$f'(0) = 2 \quad \text{and} \quad g'(0) = 1.$$  

For this example it should be clear that:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2.$$  

Here’s a visual that will hopefully clarify the relationship between $y = 2x$ and $y = e^{2x} - 1$. It should be clear that these two graphs are nearly identical near the origin.

![Graph of $y = e^{2x} - 1$ and $y = 2x$.](image)

**Figure 2:** $y = e^{2x} - 1$ and $y = 2x$.

**l’Hôpital’s rule:** If $f$ and $g$ are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$
Using the definition of derivative, here’s a justification of l’Hôpital’s rule.

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{g(a+h) - g(a)}
\]

A more general form of l’Hôpital’s rule is: If \( f \) and \( g \) are differentiable, \( f(a) = g(a) = 0 \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]

provided the right hand limit exists.

l’Hôpital’s rule also applies when \( f \) and \( g \) limits at \( a \) are \( \pm \infty \). Here the rule is stated as: provided \( f \) and \( g \) are differentiable:

- When \( x \to a \), both \( f \) and \( g \) go towards \( \pm \infty \), or
- When \( a = \pm \infty \) and as \( x \to a \), both \( f \) and \( g \) go towards \( \pm \infty \).

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]

where \( a = \pm \infty \), provided the right hand limit exists.

These forms are often referred to as indeterminate forms. Here’s the list:

- Indeterminate form of type 0/0.
- Indeterminate form of type \( \infty / \infty \).
- Indeterminate form of type 0 \cdot \infty.
- Indeterminate form of type \( \infty - \infty \).
- Indeterminate form of type 0^0.
- Indeterminate form of type \( \infty^0 \).
- Indeterminate form of type 1^\infty.

Here’s some general notes from Stewart’s book:

**Note 1** l’Hôpital’s rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied.

**Note 2** l’Hôpital’s rule is also valid for one-sided limits and for limits at \( \pm \) infinity.
1.2 Examples

1. For students planning to major in mathematics, please carefully look over the proof of l'Hôpital’s rule in Appendix F of Stewart’s textbook.

2. Show using l'Hôpital’s rule that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

3. Show using l'Hôpital’s rule that \( \lim_{x \to 1} \frac{\ln x}{x - 1} = 1 \).

4. Show using l'Hôpital’s rule that \( \lim_{x \to \infty} xe^{-x} = 0 \).

5. Show using l'Hôpital’s rule that \( \lim_{x \to \infty} e^{x} x^{2} = \infty \).

6. Show using l'Hôpital’s rule that \( \lim_{x \to \infty} \frac{5x + e^{-x}}{7x} = \frac{5}{7} \).

7. Don’t use l'Hôpital’s rule but show that \( \lim_{x \to \infty} \frac{x^{2} + \sin x}{x^{2}} = 1 \). Now try using l'Hôpital’s rule to see what happens.
8. Show using l'Hôpital’s rule that \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e. \)

9. Show using l'Hôpital’s rule that \( \lim_{x \to 0} \frac{1 - \cosh (3x)}{x} = 0. \)

10. I think these are two difficult problems, and I think everyone should give them a try.

   (a) \( \lim_{x \to 0^+} \left[ \sin \left( \frac{\pi}{2} + x \right) \right]^\frac{1}{x} = 1 \)

   (b) \( \lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x - 1} \right) = \frac{1}{2} \)

11. Marquis l'Hôpital’s used this example in his textbook.

   \[ \lim_{x \to a} \frac{\sqrt{2a^4x - x^4} - a\sqrt{a^2x}}{a - \sqrt[4]{ax^3}}, \quad a > 0 \]

   Evaluate this limit.
12. If $f'$ is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate
\[
\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}
\]

13. Given that $f(x) = x^{-x}$ where $x > 0$, and its graph.

![Figure 3: $f(x)$](image)

(a) Use l’Hôpital’s rule to explain the behavior as $x \to 0^+$.

(b) Use calculus to find the maximum value of $f$.

(c) What is $f$’s range?