1 Antiderivatives

Working backwards from the derivative can be as maddening as backing up a tractor trailer at 60 mph. In fact, as you’ll learn in MTH-122, many tricky techniques exists just to allow us to do so. And, as I hope you know, backing up while driving (or undoing differentiation) requires practice and patience. Yes, you’ll need to go slow at first, and once done you’ll need to retrace your steps forward to see if all is going as planned! Let’s proceed.

**Definition:** A function $F$ is called the antiderivative of $f$ on the interval $I$ if

$$F'(x) = f(x)$$

for all $x$ in $I$.

I am going to try to avoid the “big $F$” notation and stick with the more familiar $f$ and $f'$ only. So using the above definition again, we would now have:

**Definition:** A function $f$ is called the antiderivative of $f'$ on the interval $I$ if

$$\frac{d}{dx}[f(x)] = f'(x)$$

for all $x$ in $I$.

Here’s an example. Suppose $f'(x) = 3x^2 + 2x - 3$, find $f(x)$ (the antiderivative). Although this might be a bit frightening at first, you should realize that we’re just looking for an $f$, that when differentiated with look like $f'$. Going term-by-term we get

$$f(x) = x^3 + x^2 - 3x + C.$$  

Yes, you should note the $C$, and you should also check to verify that differentiating $f$ will in fact give you $f'$. Let’s see.

$$f(x) = x^3 + x^2 - 3x + C$$

$$f'(x) = 3x^2 + 2x - 3$$

Yes, all is well! Here’s the reason for the $C$.

**Theorem:** If $f$ is the antiderivative of $f'$ on the interval $I$, then the most general antiderivative of $f'$ on $I$ is

$$f(x) + C,$$

where $C$ is an arbitrary constant. The $C$, as you should know, will always differentiate to zero!

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1This document was prepared by Ron Bannon using \LaTeX\ 2ε.
Table 1: Table of General Antiderivatives

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n, n \neq -1$</td>
<td>$\frac{x^{n+1}}{n+1} + C$</td>
</tr>
<tr>
<td>$x^{-1}$</td>
<td>$\ln</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x + C$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x + C$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$- \cos x + C$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\tan x + C$</td>
</tr>
<tr>
<td>$\sec x \tan x$</td>
<td>$\sec x + C$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin x + C$</td>
</tr>
<tr>
<td>$\frac{1}{1+x^2}$</td>
<td>$\arctan x + C$</td>
</tr>
</tbody>
</table>

1.1 Checking the Antiderivatives

I will discuss each of these antiderivatives in class. I’d like to believe that you are already familiar with most of these (you may have to review the prior sheets though). However, I don’t care how trivial the problem is, you must always check yourself.

To check each of these antiderivatives we just need to differentiate and see what it gives back. Again, we’ll discuss these in class.

$$x^n = \frac{d}{dx} \left[ \frac{x^{n+1}}{n+1} + C \right], \ n \neq -1 \quad (1)$$

$$x^{-1} = \frac{d}{dx} [\ln |x| + C] \quad (2)$$

$$e^x = \frac{d}{dx} [e^x + C] \quad (3)$$

$$\cos x = \frac{d}{dx} [\sin x + C] \quad (4)$$

$$\sin x = \frac{d}{dx} [-\cos x + C] \quad (5)$$

$$\sec^2 x = \frac{d}{dx} [\tan x + C] \quad (6)$$

$$\sec x \tan x = \frac{d}{dx} [\sec x + C] \quad (7)$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx} [\arcsin x + C] \quad (8)$$

$$\frac{1}{1+x^2} = \frac{d}{dx} [\arctan x + C] \quad (9)$$

Certainly we will differentiate to check out antiderivatives, but I will say that (2) is a bit tricky.
To see why we need to compute the derivative of $y = \ln |x|$. Recall that using definition of the absolute value function will result in following breakdown:

$$
y = \ln |x| = \begin{cases} 
\ln x & \text{if } x > 0 \\
\ln (-x) & \text{if } x < 0 
\end{cases}
$$

You should notice that zero is excluded. Now let’s differentiate.

$$
y' = \frac{d}{dx} (\ln |x|) = \begin{cases} 
\frac{1}{x} & \text{if } x > 0 \\
\frac{1}{x} & \text{if } x < 0 
\end{cases}
$$

2 Examples

1. Find $f$, where $f'(x) = \cos x - 5e^x$.

2. Find $f$, where $f'(x) = 6/x^7$.

3. Find $f$, where $f'(x) = \sqrt[5]{x^3} - \sin x$.

4. Find $f$, where $f'(x) = (2x - 5)^2$.

5. Find $f$, where $f''(x) = 6x + 12x^2$.

6. Find $f$, where $f'''(x) = 5x + \cos x$. 


7. Find \( f \), where \( f'(x) = 1 - 6x \) and \( f(0) = 7 \).

8. Find \( f \), where \( f''(x) = 2e^x + 3 \sin x \), \( f(0) = 0 \), and \( f(\pi) = 0 \).

9. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

\[
a = \begin{cases} 
9 - 0.9t & \text{if } 0 \leq t \leq 10 \\
0 & \text{if } t > 10 
\end{cases}.
\]

If the raindrop is initially 500 m above the ground, how long does it take to fall?
I think the last problem is difficult and we’ll discuss it in class. Here’s the solution.

**Solution:** It *clearly* states that the initial velocity is downward, so

\[ v(0) = -10. \]

Furthermore, the acceleration function given is also downward,\(^2\) so

\[ a(t) = \begin{cases} 0.9t - 9 & \text{if } 0 \leq t \leq 10 \\ 0 & \text{if } t > 10 \end{cases}. \]

Now we need to work backwards to find both the velocity function, \(v(t)\), and the position function, \(s(t)\). Working from acceleration to velocity first.

\[
\begin{align*}
  a(t) &= v'(t) = 0.9t - 9 \\
  v(t) &= 0.45t^2 - 9t + v_0 \\
  v(0) &= -10 = v_0 \\
  v(t) &= 0.45t^2 - 9t - 10 
\end{align*}
\]

Then from velocity to position next.

\[
\begin{align*}
  v(t) &= s'(t) = 0.45t^2 - 9t - 10 \\
  s(t) &= 0.15t^3 - 4.5t^2 - 10t + s_0 \\
  s(0) &= s_0 = 500 \\
  s(t) &= 0.15t^3 - 4.5t^2 - 10t + 500 
\end{align*}
\]

When \(t = 10\) the raindrop reaches its terminal velocity and it is above the ground and its terminal velocity is

\[ v(10) = 45 - 90 - 10 = -55 \text{ m/sec}. \]

Now, finally, at this constant rate it will take an additional 1.8 seconds to reach the ground. So the total time for the raindrop to fall 500 meters using this model is \(11.8 \text{ seconds}\).

\(^2\)This can cause great confusion, but the book does this all the time.
**Visual Note:** Here’s the position graph.

![Position Graph](image)

Figure 1: The vertical axis is position (meters) and the horizontal is time (seconds).

The curve to the left of the red point is $s(t) = 0.15t^3 - 4.5t^2 - 10t + 500$ and to the right is $s(t) = -55(x - 10) + 100$.

Here’s the velocity graph.

![Velocity Graph](image)

Figure 2: The vertical axis is velocity (m/s) and the horizontal is time (seconds).

The curve to the left of the red point is $v(t) = 0.45t^2 - 9t - 10$ and to the right is $v(t) = -55$ until it hits the ground at 11.8 seconds, where the velocity becomes zero.