Given \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \), answer the following questions.

**Solution:** You’ll need the first derivative

\[
f'(x) = 12x^3 - 12x^2 - 24x = 12x(x + 1)(x - 2),
\]

and where it changes sign, clearly at \( x = -1, x = 0 \) and \( x = 2 \).

\[
f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2),
\]

and where it changes sign, clearly at \( x = \frac{1 \pm \sqrt{7}}{3} \).

(a) **5 points** Find the interval(s) where \( f \) is increasing.

**Solution:** By sign analysis, \((-1, 0), (2, \infty)\).

(b) **5 points** Find the interval(s) where \( f \) is decreasing.

**Solution:** By sign analysis, \((-\infty, -1), (0, 2)\).

(c) **6 points** Find the interval(s) where \( f \) is concave-up.

**Solution:** By sign analysis, \((-\infty, \frac{1 - \sqrt{7}}{3}), \left(\frac{1 + \sqrt{7}}{3}, \infty\right)\).

(d) **6 points** Find the interval(s) where \( f \) is concave-down.

**Solution:** By sign analysis, \(\left(\frac{1 - \sqrt{7}}{3}, \frac{1 + \sqrt{7}}{3}\right)\).
2. Given that

\[
\begin{align*}
  f(x) &= \frac{x^3 - x^2 - 1}{x^2 + 1} \\
  f'(x) &= \frac{x^2 (x^2 + 3)}{(x^2 + 1)^2} \\
  f''(x) &= \frac{2x (3 - x^2)}{(x^2 + 1)^3}
\end{align*}
\]

and the following partial graph of \(f(x)\).

![Partial graph of \(f(x)\).](image)

Figure 1: Partial graph of \(f(x)\).

Answer each of the following questions.

(a) [5 points] The domain of \(f(x)\).

**Solution:** No troubles here, so the domain is \(\mathbb{R}\).

(b) [5 points] The \(y\)-intercept.

**Solution:** Just set \(x = 0\) and you’ll get \((0, -1)\).

(c) [3 points] The equation of the slant asymptote.

**Solution:** You’ll need to do the *dreaded long division*, and then you’ll find that the slant asymptote is \(y = x - 1\).

(d) [4 points] Find all inflection points on \(f(x)\). It’s hard to see them on the graph, but they’re there.

**Solution:** The second derivative

\[
    f''(x) = \frac{2x (3 - x^2)}{(x^2 + 1)^3},
\]
is changing signs at \( x = 0, x = \pm \sqrt{3} \). So the points of inflection are

\[
\left(-\sqrt{3}, f\left(-\sqrt{3}\right)\right), (0, f(0)), \left(\sqrt{3}, f\left(\sqrt{3}\right)\right)
\]

or if you prefer

\[
\left(-\sqrt{3}, -\frac{3\sqrt{3} + 4}{4}\right), (0, -1), \left(\sqrt{3}, \frac{3\sqrt{3} - 4}{4}\right)
\]

3. **10 points** Find the dimensions of the rectangle of largest area that has its base on the \( x \)-axis and its other two vertices above the \( x \)-axis and lying on the parabola \( y = 8 - x^2 \).

**Solution:** If we let \( x > 0 \) represent the distance from the origin to the bottom right corner of our rectangle, the base of the rectangle is \( 2x \) and its height is \( 8 - x^2 \), the area is \( A = f(x) = 2x (8 - x^2) \). Furthermore, it should be obvious that \( 0 < x < 2\sqrt{2} \).

\[
\begin{align*}
  f(x) &= 2x (8 - x^2) \\
  f(x) &= 16x - 2x^3 \\
  f'(x) &= 16 - 6x^2
\end{align*}
\]

The critical numbers are

\[
16 - 6x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{4}{\sqrt{6}}.
\]

Since \( 0 < x < 2\sqrt{2} \) we only need to analyze the positive critical number on this interval. Using sign analysis on this open interval we find the that \( f' \) is positive to the left of \( \frac{4}{\sqrt{6}} \), and \( f' \) is negative to the right of \( \frac{4}{\sqrt{6}} \). That is, the area function is increasing to the left of \( \frac{4}{\sqrt{6}} \) and decreasing to the right of \( \frac{4}{\sqrt{6}} \). So the area is maximized at \( x = \frac{4}{\sqrt{6}} \).

The dimensions are

\[
\frac{8}{\sqrt{6}} \times \frac{16}{3}
\]
Here’s a visual of this word problem.

Figure 2: Partial graph of $y = 8 - x^2$ and the rectangle.

4. Given that

$$ f(x) = \sqrt{\frac{x^2 + 1}{x + 1}} $$

$$ f'(x) = \frac{x^2 + 2x - 1}{2(x + 1)\sqrt{(x^2 + 1)(x + 1)}} $$

and the following partial graph of $f(x)$. Answer each of the following questions.

Figure 3: Partial graph of $f(x)$.

(a) [4 points] The domain of $f(x)$.

Solution: Here, although obvious, we must have

$$ \frac{x^2 + 1}{x + 1} \geq 0 \quad \Rightarrow \quad x \in (-1, \infty), $$

and this is certainly indicated by the partial graph of $f(x)$. 

(b) 4 points The point form of the y-intercept.

**Solution:** Just set $x = 0$ and you’ll get $(0, 1)$.

(c) 4 points The equation of the vertical asymptote.

**Solution:** As $x \to -1^+$ we have $f(x) \to \infty$ so the vertical asymptote is $x = -1$.

(d) 4 points Evaluate $\lim_{x \to \infty} f(x)$.

**Solution:**

$$\lim_{x \to \infty} f(x) = \infty$$

(e) 5 points The global minimum point on $f(x)$.

**Solution:** Here you will need to find the critical numbers in the domain of $f$. Looking at the derivative, the only possibility is where the numerator is zero.

$$x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x = -1 \pm \sqrt{2}$$

Only $-1 + \sqrt{2}$ is in the domain. It should be further noted that $f$ is decreasing on $(-1, -1 + \sqrt{2})$ and is increasing on $(-1 + \sqrt{2}, \infty)$, therefore we have a minimum at

$$\left( \sqrt{2} - 1, f\left(\sqrt{2} - 1\right) \right)$$

or if you, or your calculator, does the work,

$$\left( \sqrt{2} - 1, \sqrt{2\sqrt{2} - 2} \right)$$

(f) 8 points The range of $f(x)$.

**Solution:** Using the work from above, we have

$$\left[f(\sqrt{2} - 1), \infty\right)$$

or if you, or your calculator, does the work,

$$\left[\sqrt{2\sqrt{2} - 2}, \infty\right)$$