1. 

Evaluate \( \int_0^1 x^2 (1 + 2x^3)^5 \, dx \).

**Solution:** Use \( u \)-substitution with \( u = 1 + 2x^3 \) and \( du = 6x^2 \, dx \).

\[
\int_0^1 x^2 (1 + 2x^3)^5 \, dx = \frac{1}{6} \int_1^3 u^5 \, du
\]

\[
= \frac{u^6}{36} \bigg|_1^3
\]

\[
= \frac{182}{9}
\]
2. **7 points** Find the volume of the solid obtained by rotating the region bounded by 
\[ y = \sqrt{x}, \quad y = 2, \quad \text{and} \quad x = 0 \] about the \( y \)-axis.

**Solution:** Just set-up the integral and evaluate.

\[
\int_0^2 \pi y^4 \, dy = \frac{\pi y^5}{5} \bigg|_0^2 = \frac{32\pi}{5}
\]
3. 7 points Evaluate the limit, if it exists.

\[ \lim_{x \to 0} \left( \frac{1}{x \sqrt{1 + x}} - \frac{1}{x} \right) \]

Solution:

\[
\lim_{x \to 0} \left( \frac{1}{x \sqrt{1 + x}} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{1 - \sqrt{1 + x}}{x \sqrt{1 + x}} \cdot \frac{1 + \sqrt{1 + x}}{1 + \sqrt{1 + x}}
\]

\[
= \lim_{x \to 0} \frac{-x}{x \sqrt{1 + x} (1 + \sqrt{1 + x})}
\]

\[
= \lim_{x \to 0} \frac{-1}{\sqrt{1 + x} (1 + \sqrt{1 + x})} = \frac{-1}{2}
\]
4. [7 points] Use the Fundamental Theorem of Calculus to find the derivative of the function defined by

\[ g(x) = \int_{5}^{x^3} t^2 \sin t \, dt. \]

**Solution:**

\[
\frac{d}{dx} [g(x)] = \frac{d}{dx} \left[ \int_{5}^{x^3} t^2 \sin t \, dt \right]
\]

\[
g'(x) = x^6 \sin x^3 \cdot 3x^2
\]

\[
g'(x) = 3x^8 \sin x^3
\]
5. 7 points Find the area of the region that lies to the right of the $y$-axis and to the left of the parabola $x = 5y - y^2$.

Solution:

$$\int_{0}^{5} (5y - y^2) \, dy = \left[ \frac{5y^2}{2} - \frac{y^3}{3} \right]_{0}^{5}$$

$$= \left( \frac{125}{2} - \frac{125}{3} \right) - \left( \frac{0}{2} - \frac{0}{3} \right)$$

$$= \frac{125}{6}$$
6. **7 points** If $f$ is differentiable at $a$, where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}.$$

**Solution:** It is an easy problem, I hope you agree.

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} = \lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x + a} \cdot (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x + a} \cdot \lim_{x \to a} (\sqrt{x} + \sqrt{a})$$

$$= f'(a) \cdot 2\sqrt{a}$$

$$= 2\sqrt{a} \cdot f'(a)$$
7. **7 points** Suppose $f(x) = e^x(ax + b)$, where $a$ and $b$ are unknown, but you are given that the tangent line to the graph of $f$ at $x = 1$ has equation $y = 7x - 8$. Find $a$ and $b$. As a visual aid, I am graphing both $f(x)$ with the proper values of $a$ and $b$, and $y = 7x - 8$.

![Figure 1: Partial graph of $f(x)$ and the line tangent to $f$ at $x = 1$.]

**Solution:** First the derivative of $f(x)$.

$$f(x) = e^x(ax + b)$$
$$f'(x) = e^x(ax + b) + ae^x$$

Given that $y = 7x - 8$ is tangent to $f$ at $x = 1$, we know the value of $f$ at $x = 1$ is $y = 7 \cdot 1 - 8 = -1$. That is,

$$f(1) = e(a + b) = ea + eb = -1 \quad \Rightarrow \quad a + b = -\frac{1}{e}$$

Furthermore the slope of this tangent line at $x = 1$ is 7, so

$$f'(1) = e(a + b) + ae = 2ae + be = 7 \quad \Rightarrow \quad 2a + b = \frac{7}{e}$$

Using the method of elimination with the above two linear equations, we find that

$$a = \frac{8}{e} \quad \text{and} \quad b = -\frac{9}{e}$$