MSTM 6033 — Fall — 2004  
Teachers College — Columbia University  
Assignment # 2 — September 24, 2004

Name: ____________________________________________________________

Signature: _________________________________________________________


1. Problem 18, §1.3 — Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} \), \( \mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix} \). For what value(s) of \( h \) is \( \mathbf{y} \) in the plane generated by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \)?

Solution: All linear combinations of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) will span the plane that includes these two vectors, if we want to find the linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) that produces \( \mathbf{y} \), we will need to solve \( a \mathbf{v}_1 + b \mathbf{v}_2 = \mathbf{y} \). In matrix form: 

\[
\begin{bmatrix}
1 & -3 \\
0 & 1 \\
-2 & 8
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
=
\begin{bmatrix}
h \\
-5 \\
-3
\end{bmatrix}
\]

which can be thought of as a simple system of linear equations:

\[
\begin{align*}
a - 3b &= h \\
b &= -5 \\
-2a + 8b &= -3
\end{align*}
\]

Where the solution for \( b \) can be obtained by inspection, \( b = -5 \). Substituting \( b = -5 \) into 
\(-2a + 8b = -3\) yields a solution for \( a \), where \( a = -\frac{37}{2} \). Finally, substituting \( b = -5 \) and 
\( a = -\frac{37}{2} \) into \( a - 3b = h \), gives the value of \( h = -\frac{7}{2} \).

2. Problem 1, §1.4 — Compute the product, using (a) the definition, as in Example 1, and (b) the row-vector rule for computing \( \mathbf{A}\mathbf{x} \). If the product is undefined, explain why.

\[
\begin{bmatrix}
-4 & 2 \\
1 & 6 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
-2 \\
7
\end{bmatrix}
\]

Solution: The product is not defined. We would need another column of \( \mathbf{A} \), or one less.
row of \( \mathbf{b} \). The number columns in \( A \) must match the number of rows in \( B \) for the product \( AB \) to be defined.

3. Problem 12, §1.4 — Write the augmented matrix for the linear system that corresponds to the matrix equation \( A\mathbf{x} = \mathbf{b} \). Then solve the system and write the solution as a vector.

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
-3 & -1 & 2 \\
0 & 5 & 3
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}
\]

Solution: Augmented matrix form of these matrices is:

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
-3 & -1 & 2 & 1 \\
0 & 5 & 3 & -1
\end{bmatrix}
\]

Elementary row operations, in order given:

\[
3R_1 + R_2 \rightarrow R_2 \\
-R_2 + R_3 \rightarrow R_3 \\
-\frac{1}{2}R_3 \rightarrow R_3
\]

Produces:

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 5 & 3 & -1
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 & 0 \\
0 & 5 & 3 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Simple back-substitution gives the solution: \( x_3 = 1 \), \( x_2 = -\frac{4}{5} \) and \( x_1 = \frac{3}{5} \). In vector form, the solution is:

\[
\begin{bmatrix}
3 \\
-4 \\
5
\end{bmatrix}
\]

4. Problem 13, §1.4 — Let \( \mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \), \( A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix} \). Is \( \mathbf{u} \) in the plane in \( \mathbb{R}^3 \) spanned by the columns of \( A \)? Why or why not?

Solution: The columns of \( A \) are linearly independent in \( \mathbb{R}^3 \) and span a plane in this space. To see if \( \mathbf{u} \) is in the span of \( A \), that is a linear combination of the columns of \( A \), Solve

\[
x_1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix},
\]

for \( x_1 \) and \( x_2 \). Using simple matrix algebra yields a solution \( x_1 = \frac{5}{2} \) and \( x_2 = \frac{3}{2} \). Therefore \( \mathbf{u} \) is in the plane spanned by the column space of \( A \).
5. Problem 15, §1.4 — Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible $b$, and describe the set of all $b$ for which $Ax = b$ does have a solution.

Solution: Augmented matrix form of $Ax = b$ is:

\[
\begin{bmatrix}
2 & -1 & | & b_1 \\
-6 & 3 & | & b_2
\end{bmatrix}
\]

Elementary row operations produces:

\[
\begin{bmatrix}
2 & -1 & | & b_1 \\
0 & 0 & | & 3b_1 + b_2
\end{bmatrix}
\]

The system has a solution when $3b_1 + b_2 = 0$ only, so a solution only exists when $b_2 = -3b_1$. In vector form $x = \begin{bmatrix} b_1 + k \\ 2k \end{bmatrix}$, where $k \in \mathbb{R}$.

6. Problem 34, §1.4 — Suppose $A$ is a $3 \times 3$ matrix and $b$ is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of $A$ must span $\mathbb{R}^3$.

Solution: Using the following:

Theorem$^2$: Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent.

(a) The columns of $A$ form a basis for $\mathbb{R}^n$.
(b) The equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.
(c) $A$ is an invertible matrix.
(d) The determinant of $A$ is nonzero.
(e) $A$ is row equivalent to the identity matrix.

We are told that $Ax = b$ has a unique solution which is entry (b) in the above theorem. Therefore, statement (a) is implied. Of course, if we have a basis for $\mathbb{R}^3$, the column space spans $\mathbb{R}^3$. Q.E.D.

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