MSTM 6033 — Fall — 2004

Teachers College — Columbia University

Assignment # 8[1] — November 23, 2004

Name: ______________________________

Signature: ______________________________

Assigned homework from David C. Lay's textbook: *Linear Algebra and Its Application*, Third Edition, Addison-Wesley, 2003, ISBN: 0-201-70970-8.

1. Here's a list (in alphabetical order) of some of the topics we've discussed. Suppose you were teaching a class on the subject; make an outline or chart showing in what order you would present them, or how they are connected: basic variables, basis, column space, determinant, dimension, echelon form, eigenvalues and eigenfunctions, free variables, linear equations, linear independence, linear transformation, matrix invert-ability, matrix notation, matrix rank, matrix row space, nullspace, row operations, spanning set, subspace, vector spaces.

   Answer: If I were faced with teaching a course in linear algebra, I'd certainly be a bit nervous, and rightfully so. My teaching has mainly focus on remedial education with occasional forays into developmental mathematics including integral calculus. Even at the developmental level, I mainly deal with intuitive concepts that are easy for me to teach, however, I believe linear algebra is really the first standard mathematics course where abstract ideas have little intuitive basis. I recall from my own studies that linear algebra, along with proof level analysis were the dividing lines in most people's mathematics education.

   Certainly, if forced, I think I could teach an introductory linear algebra course, but I'd rely heavily on the text and would watch Gilbert Strang's web based videos[2] before lecturing.

   Outline of Topics

   Part 1: The Geometry of Simple Linear Equations[3]

   - **linear equations** – using a simple 2×2 linear system to review basic Gaussian elimination.
   - **matrix notation** – using the same 2×2 linear system to review basic Gaussian elimination in matrix form.

[1]This document was prepared by Ron Bannon using LaTeX. Source and pdf are available by emailing a request to `rbannon@mac.com`.

[2]http://ocw.mit.edu/OcwWeb/Mathematics/18-06Linear-AlgebraFall2002/VideoLectures/index.htm

[3]Problems with only one point in the solution space.

- **matrix row space** – examine the row space of a 3×3 linear system and how it's related to Gaussian elimination.
- **row operations** – formally describe the row operations involved in Gaussian elimination for a 3×3 linear system.
- **echelon form** – formally state the outcome of Gaussian elimination and how it relates to echelon form for a $A\mathbf{x} = \mathbf{b}$.

Part 2: The Geometry of Linear Equations[4]

- **basic variables** – using a simple 3×3 linear system to review basic Gaussian elimination where the solution space is a plane. Basic variables will be defined here.
- **free variables** – using the same 3×3 linear system to define free variables.
- **vector space** – Using linear combination of vectors to see how they define a space.
- **subspace** – Using the same 3×3 linear system to describe the solution as a subspace of $\mathbb{R}^3$. Furthermore, formally define what it means to be a subspace for vectors.
- **spanning set** – Describe the solution space as a set of points that are 'spanned' by vectors.

Part 3: Column Space and Nullspace

- **column space** – Rethink the $A\mathbf{x} = \mathbf{b}$ in terms of combinations of columns instead of using the more intuitive row space.
- **nullspace** – Describe a fundamental subspace and how it relates to the row and column space.
- **matrix invert-ability** – Finally describe the conditions necessary for a matrix to be invertible.

Part 4: Solving $A\mathbf{x} = \mathbf{b}$: Row Reduced Form.

- **matrix rank** – Pull together the concepts of subspaces and the rank of a matrix.

Lecture 5: Independence, Basis, and Dimension

- **linear independence** – Take a look at a set of vectors that span a space to see if there's redundancy.
- **basis** – Talk about set of vectors that span a space where there's no redundancy.
- **dimension** – Relate the concept of a minimum sized spanning set, or basis, to the dimension of the space.

Part 6: Properties of Determinants

- **determinant** – How determinants are related to solving equations and invertibility.
- **eigenvalues and eigenfunctions** – Using determinants to find eigenvalues and eigenvectors.

[4]Problems with more than one point, or no points in the solution space.

Part 7 : Linear Transformations and Their Matrices

- **linear transformation** – defining what a linear transformation is.

2. Problem 15, §5.1 — Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \qquad \lambda = 3$$

Solution: Solving for $\mathbf{v}$:

$$\begin{aligned} A\mathbf{v} &= 3\mathbf{v} \\ \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= 3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

This results (by inspection) is an equation with two free variables $v_1 + 2v_2 + 3v_3 = 0$. So a basis for eigenvalue $\lambda = 3$ is:

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$

3. Problem 10, §5.2 — Find the characteristic polynomial of the matrix, $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$,using either cofactor expansion or the special formula for 3×3 determinants described prior to Exercises 15–18 in §3.1. [*Note*: Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

Solution: Let $A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$. The characteristic equation is a function of λ, where $f(\lambda) = \det(A - \lambda I)$.

$$\begin{aligned}
A - \lambda I &= \begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix} \\
\det(A - \lambda I) &= \begin{vmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} \\
\det(A - \lambda I) &= -\lambda \cdot \begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3 \cdot \begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix} \\
\det(A - \lambda I) &= -\lambda \cdot \left(\lambda^2 - 4\right) - 3 \cdot (-3\lambda - 2) + 1 \cdot (6 + \lambda) \\
\det(A - \lambda I) &= \boxed{-\lambda^3 + 14\lambda + 12}
\end{aligned}$$

4. Problem 2, §5.3 — Let $A = PDP^{-1}$ and compute A^4.

$$P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix}$$

Solution:

A^4 can be computed directly, by direct multiplication.

$$\begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix}^4 = \begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix} \begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix} \begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix} \begin{bmatrix} 5.5 & 3 \\ -7.5 & -4 \end{bmatrix}$$

However it is simpler, to compute A^4 by using A's given diagonal matrix representation. This would be especially true if the power were more difficult to compute.

$$\begin{aligned}
A &= PDP^{-1} \\
A^4 &= \left(PDP^{-1}\right)^4 \\
A^4 &= PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1} \\
A^4 &= PDIDIDIDP^{-1} \\
A^4 &= PDDDDP^{-1} \\
A^4 &= PD^4P^{-1}
\end{aligned}$$

$$\begin{aligned}
A^4 &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^4 \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}^{-1} \\
A^4 &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \\
A^4 &= \boxed{\frac{1}{16} \begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}}
\end{aligned}$$

5. Problem 10, §5.3 — Diagonalize the matrix, if possible.

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, where A is diagonalizable if and only if there are enough eigenvectors to form a basis for $\mathbb{R}^2$. We first must find the eigenvalues of the matrix A.

$$\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\
A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\
(A - \lambda I)\,\mathbf{x} &= \mathbf{0}
\end{aligned}$$

So $A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$, where we need to find the scalars λ such that $A - \lambda I$ is not invertible (we don't want the trivial solution where $\mathbf{x} = \mathbf{0}$). We need to find the $\det(A - \lambda I)$ as a function of λ, *i.e.* $f(\lambda) = \det(A - \lambda I)$.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} \\
\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} &= (2-\lambda)(1-\lambda) - 3 \cdot 4 \\
(2-\lambda)(1-\lambda) - 3 \cdot 4 &= \lambda^2 - 3\lambda - 10
\end{aligned}$$

Solving $f(\lambda) = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5) = 0$ for λ gives two solutions: $\lambda_1 = -2$ and $\lambda_2 = 5$. Now, using these two eigenvalues we need to find their corresponding eigenvectors.

Solving for $\mathbf{v}_1$:

$$\begin{aligned} A\mathbf{v}_1 &= -2\mathbf{v}_1 \\ \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} &= -2 \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} \\ \mathbf{v}_1 &= k \begin{bmatrix} 3 \\ -4 \end{bmatrix}. \end{aligned}$$

Solving for $\mathbf{v}_2$:

$$\begin{aligned} A\mathbf{v}_2 &= 5\mathbf{v}_2 \\ \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} &= 5 \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} \\ \mathbf{v}_2 &= k \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Finally:

$$P = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Where:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \boxed{PDP^{-1} = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}}.$$