MTH 121 — Fall — 2004

Essex County College — Division of Mathematics

Test # 2[1] — Created December 6, 2004

Name: ________________________________

Signature: ________________________________

Show all work *clearly* and in *order*, and box your final answers. Justify your answers algebraically whenever possible. You have at most 80 minutes to take this 100 point exam. No cellular phones allowed.

1. (10 points) — Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $-\frac{1}{2} \le x \le 4$.

   Solution: Using the **Closed Interval Method**.

   (a) Find the values of $f(x)$ at the critical numbers of $f(x)$ in the open interval $\left(-\frac{1}{2}, 4\right)$. Since $f'(x) = 3x^2 - 6x$ the critical numbers are $x = 0$ and $x = 2$. Evaluating $f(0) = 1$ and $f(2) = -3$.

   (b) Find the values of $f(x)$ at the endpoints of the closed interval $\left[-\frac{1}{2}, 4\right]$. Evaluating $f\left(-\frac{1}{2}\right) = \frac{1}{8}$ and $f(4) = 17$.

   (c) The largest value of $f(x)$ from the above two steps is the absolute maximum value; the smallest these values is the absolute minimum.

   Answer: The absolute maximum is: $f(4) = 17$; and the absolute minimum is $f(2) = -3$.

2. (15 points) — Given $f(x) = x^2 + 5x + 9$, verify the following:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8}{n} \cdot f\left(-3 + \frac{8i}{n}\right)\right] = \frac{488}{3}$$

   Solution: First evaluate $f\left(-3 + \frac{8i}{n}\right)$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8}{n} \cdot f\left(-3 + \frac{8i}{n}\right)\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8}{n} \cdot \left(\frac{64i^2 - 8ni + 3n^2}{n^2}\right)\right]$$

   Next, simplify the summand.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8}{n} \cdot \left(\frac{64i^2 - 8ni + 3n^2}{n^2}\right)\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{512}{n^3} \cdot i^2 - \frac{64}{n^2} \cdot i + \frac{24}{n}\right]$$

[1] This document was prepared by Ron Bannon using LaTeX.

Next, evaluate the sum.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{512}{n^3}\cdot i^2-\frac{64}{n^2}\cdot i+\frac{24}{n}\right]=\lim_{n\to\infty}\left[\frac{512}{n^3}\cdot\frac{n\left(n+1\right)\left(2n+1\right)}{6}-\frac{64}{n^2}\cdot\frac{n\left(n+1\right)}{2}+\frac{24}{n}\cdot n\right]$$

Next, simplify and evaluate the limit.

$$\lim_{n\to\infty}\left[\frac{256}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)-32\left(1+\frac{1}{n}\right)+24\right]=\frac{256}{3}\cdot 2-32+24=\frac{488}{3}$$

Q.E.D.

3. (15 points) — A cylindrical can is to be made to hold one liter of oil. Find the dimension that will minimize the cost of the metal to manufacture the can.[2]

Solution: The cost, assuming the can's top, bottom, and side are made from the same material, is equivalent to minimizing the surface area. The can's surface area is:

$$S=2\pi r^2+2\pi rh$$

Unfortunately there's two variables and we to find a relationship to rewrite S in terms of either r or h only. Using $V=1,000=\pi r^2h$, and solving for $h=\frac{1,000}{\pi r^2}$, then substituting into $S=2\pi r^2+2\pi rh$, yields:

$$S\left(r\right)=2\pi r^2+2\pi r\left(\frac{1,000}{\pi r^2}\right)=2\pi r^2+2,000r^{-1}$$

The variable r is contained on the open interval $(0,\infty)$, and we just need to find the critical numbers by taking the derivative of $S\left(r\right)$, and solving $S'\left(r\right)=0$ or $S'\left(r\right)=\text{DNE}$.

$$S'\left(r\right)=4\pi r-2,000r^{-2}$$

There's only one critical number contained in the open interval $(0,\infty)$, $r=\sqrt[3]{\frac{500}{\pi}}$. Where $S'\left(r\right)<0$ for $r<\sqrt[3]{\frac{500}{\pi}}$ and $S'\left(r\right)>0$ for $r>\sqrt[3]{\frac{500}{\pi}}$. Hence, $r=\sqrt[3]{\frac{500}{\pi}}$ minimizes $S\left(r\right)$.

To find h, just substitute $r=\sqrt[3]{\frac{500}{\pi}}$ into $h=\frac{1,000}{\pi r^2}$ and simplify.

$$h=\frac{1,000}{\pi}\left(\sqrt[3]{\frac{\pi}{500}}\right)^2=10\sqrt[3]{\frac{4}{\pi}}$$

So, the dimensions are:

- $r=\sqrt[3]{\frac{500}{\pi}}=5\sqrt[3]{\frac{4}{\pi}}\approx 5.42$ centimeters, and
- $h=10\sqrt[3]{\frac{4}{\pi}}\approx 10.84$ centimeters.

[2] You'll need to find the radius and height. Formulas that might be helpful: volume of a cylinder is given by $V=\pi r^2h$; circumference of a circle is given by $C=2\pi r$; area of a circle is given by $A_c=\pi r^2$; and area of a rectangle is given by $A_r=lw$. You should also be aware that 1 L = 1,000 cm^3.

4. (10 points) — Find $f(x)$ if $f''(x) = 2 + \cos x$, $f(0) = -1$, and $f\left(\frac{\pi}{2}\right) = 0$.

Solution:

$$\begin{aligned} f'(x) &= 2x + \sin x + C_1 \\ f(x) &= x^2 - \cos x + C_1 x + C_2 \end{aligned}$$

Using $f(0) = -1$ to solve for C_2.

$$\begin{aligned} f(0) &= 0^2 - \cos 0 + C_1 0 + C_2 = -1 \\ -1 + C_2 &= -1 \\ C_2 &= 0 \end{aligned}$$

Using $f\left(\frac{\pi}{2}\right) = 0$, $C_2 = 0$ to solve for C_1.

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \left(\frac{\pi}{2}\right)^2 - \cos\left(\frac{\pi}{2}\right) + C_1\left(\frac{\pi}{2}\right) = 0 \\ \frac{\pi^2}{4} + C_1\left(\frac{\pi}{2}\right) &= 0 \\ C_1\left(\frac{\pi}{2}\right) &= -\frac{\pi^2}{4} \\ C_1 &= -\frac{2}{\pi} \cdot \frac{\pi^2}{4} \\ C_1 &= -\frac{\pi}{2} \end{aligned}$$

So, $\boxed{f(x) = x^2 - \cos x - \frac{\pi}{2}x}$.

5. (50 points total) — Given:

$$\begin{aligned} f(x) &= \frac{x^2 + 7x + 3}{x^2} = 1 + \frac{7}{x} + \frac{3}{x^2} \\ f'(x) &= -\frac{7x + 6}{x^3} \\ f''(x) &= \frac{14x + 18}{x^4} \end{aligned}$$

Answer the following questions.

(a) (6 points) — x-intercept(s): Answer: $\boxed{\left(\frac{-7-\sqrt{37}}{2}, 0\right); \left(\frac{-7+\sqrt{37}}{2}, 0\right)}$

(b) (3 points) — y-intercept(s): Answer: $\boxed{\text{none}}$

(c) (3 points) — vertical asymptote(s): Answer: $\boxed{x = 0}$

(d) (4 points) — horizontal asymptote(s): Answer: $\boxed{y = 1}$

(e) (4 points) — domain: Answer: $\boxed{\mathbb{R}, x \neq 0}$

(f) (5 points) — range: Answer: $\boxed{\left[-\frac{37}{12}, \infty\right)}$

(g) (4 points) — local maximum(s): Answer: $\boxed{\text{none}}$

(h) (5 points) — local minimum(s): Answer: $\boxed{\left(-\frac{6}{7}, -\frac{37}{12}\right)}$

(i) (4 points) — global maximum(s): Answer: $\boxed{\text{none}}$

(j) (5 points) — global minimum(s): Answer: $\boxed{\left(-\frac{6}{7}, -\frac{37}{12}\right)}$

(k) (7 points) — point(s) of inflection: Answer: $\boxed{\left(-\frac{9}{7}, -\frac{71}{27}\right)}$