2007 A. Shloming Mathematics Prize Examination
Essex County College—Division of Mathematics and Physics
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Name: _____

Signature: _____

If the question has choices, select one answer; if the question is open ended, write your final answer on the line provided. **Five points** for each correct answer. No calculators are allowed, and the use of cellular phones is strictly forbidden.

1. On Monday a store put out ten watermelons to be sold, and some were sold. On Tuesday the number left over was doubled, and again the same number was sold as on Monday. On Wednesday, the number left over was tripled, and the same number sold as on Monday, leaving none left over. How many were sold each day?

   Answer: 6

   Solution: Let x be the number sold each day. At the end of each day we have:

   $$10 - x; \quad 2(10 - x) - x \Rightarrow 20 - 3x; \quad 3(20 - 3x) - x \Rightarrow 60 - 10x.$$

   Clearly, since the last day has zero watermelons left, x must be 6.

2. If the point $(x, -4)$ lies on the straight line joining the points $(0, 8)$ and $(-4, 0)$, then x is equal to

   (a) -2

   (b) 2

   (c) -8

   (d) 6

   (e) -6

   Answer: e

   Solution: Using slope.

   $$\frac{8-0}{0+4} = 2 = \frac{-4-8}{x-0} = -\frac{12}{x}, \quad \Rightarrow \quad x = -6.$$

[1]This document was prepared by Ron Bannon using LaTeX 2_ε and was based on a variety of sources.

3. The sum of 22^2 and 19^2 equals the sum of another pair of two-digit numbers squared. Find the numbers.

Answer: 13, 26

Solution:

$$\begin{aligned} 22^2 + 19^2 &= 845 \\ &= 5 \cdot 169 \\ &= 5 \cdot 13^2 \\ &= 1 \cdot 13^2 + 4 \cdot 13^2 \\ &= 13^2 + 26^2 \end{aligned}$$

4. If $a = \log_8 225$ and $b = \log_2 15$, then

(a) $a = \dfrac{b}{2}$

(b) $a = \dfrac{2b}{3}$

(c) $a = b$

(d) $b = \dfrac{a}{2}$

(e) $a = \dfrac{3b}{2}$

Answer: b

Solution:

$$a = \log_8 225 = \log_8 15^2 = 2\log_8 15 = \frac{2\log_2 15}{\log_2 8} = \frac{2b}{3}$$

5. If $f(x) = ax^2 - \sqrt{2}$, for all $a > 0$ and $f\left(f\left(\sqrt{2}\right)\right) = -\sqrt{2}$, then a is

(a) $\dfrac{2-\sqrt{2}}{2}$

(b) $\dfrac{1}{2}$

(c) $2 - \sqrt{2}$

(d) $\dfrac{\sqrt{2}}{2}$

(e) $\dfrac{2+\sqrt{2}}{2}$

Answer: d

Solution:

$$\begin{aligned} f\left(f\left(\sqrt{2}\right)\right) &= -\sqrt{2} \\ f\left(2a-\sqrt{2}\right) &= -\sqrt{2} \\ 2a^2-\sqrt{2}a-\sqrt{2} &= -\sqrt{2} \\ 2a^2-\sqrt{2}a &= 0 \\ a\left(2a-\sqrt{2}\right) &= 0 \end{aligned}$$

Since $a > 0$ we have:

$$2a-\sqrt{2}=0 \quad \Rightarrow \quad a=\frac{\sqrt{2}}{2}.$$

6. The consecutive angles of a trapezoid form an arithmetic sequence. If the smallest angle is 75°, then the largest angle is

(a) 95°

(b) 100°

(c) 105°

(d) 110°

(e) 115°

Answer: c

Solution: Let d be the common difference.

$$\begin{aligned} 360^\circ &= 75^\circ + (75^\circ + d) + (75^\circ + 2d) + (75^\circ + 3d) \\ 360^\circ &= 300^\circ + 6d \\ 60^\circ &= 6d \\ 10^\circ &= d \end{aligned}$$

So, the largest angle is:

$$75^\circ + 3d = 75^\circ + 30^\circ = 105^\circ.$$

7. A man is digging a hole by standing in it. He is 5 feet 10 inches tall. When we come upon him he tells us that he is one-forth done and that when he is finished, the top of this head will be three times as far below ground as it is now above ground. How deep will the hole be when finished?

   Answer: 160 inches or 13 feet 4 inches.

   Solution: Let x be the distance (in inches) his head is above ground and y be the depth (in inches) of the hole when we come upon him. One equation for his height (70 inches = 5 feet 10 inches) is $x+y=70$. When he's finished we have another equation, $4y = 70+3x$. Solve the system using substitution.

$$4y = 70 + 3\,(70 - y) \quad \Rightarrow \quad y = 40$$

8. What is the integer n for which

$$5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}?$$

   Answer: 24

   Solution:

$$\begin{aligned} 5^n + 5^n + 5^n + 5^n + 5^n &= 5^{25} \\ 5 \cdot 5^n &= 5^{25} \\ 5^{n+1} &= 5^{25} \\ n+1 &= 25 \\ n &= 24 \end{aligned}$$

9. If $f\,(x) = ax + b$ and $f^{-1}\,(x) = bx + a$ with a and b real, what is the value of $a + b$?

   Answer: -2

   Solution:

$$\begin{aligned} f\left(f^{-1}\,(x)\right) &= x \\ f\,(bx + a) &= x \\ a\,(bx + a) + b &= x \\ abx + a^2 + b &= x \end{aligned}$$

   Clearly $ab = 1$ and $a^2 + b = 0$, so

$$a^2 + b = 0 \quad \Rightarrow \quad a^2 + \frac{1}{a} = 0 \quad \Rightarrow \quad a = b = -1.$$

10. Suppose that a, b, c, d, and e are real numbers that satisfy this system of three equations:

$$\begin{cases} 13a + 26b + 2c + 13d + 3e = 18 \\ 6a + 12b + c + 6d + e = 7 \\ 5a + 10b + c + 5d + e = 6 \end{cases},$$

what is the value of e?

Answer: 3

Solution: Not easy, but you may notice that:

$$(R_1 + R_3) - 3R_2 \quad \Rightarrow \quad e = 3.$$

11. Suppose that $f(x) = ax + b$, where a and b are real numbers. Given that

$$f(f(f(x))) = 8x + 21,$$

what is $a + b$?

Answer: 5

Solution:

$$\begin{aligned} f(f(f(x))) &= 8x + 21 \\ f(f(ax + b)) &= 8x + 21 \\ f(a(ax + b) + b) &= 8x + 21 \\ a(a(ax + b) + b) + b &= 8x + 21 \\ a^3x + a^2b + ab + b &= 8x + 21 \end{aligned}$$

Clearly $a = 2$ and $a^2b + ab + b = 21$ resulting in:

$$4b + 2b + b = 21 \quad \Rightarrow \quad b = 3.$$

12. Simplify the expression

$$\left(\sqrt{2}\right)^{\log_2 9}.$$

Answer: 3

Solution:

$$\left(\sqrt{2}\right)^{\log_2 9} = \left(\sqrt{2}\right)^{2\log_2 3} = 2^{\log_2 3} = 3$$

13. An integer x, with $10 \leq x \leq 99$, is to be chosen. If all choices are equally likely, what is the probability that at least one digit of x is a 7?

Answer: $\frac{1}{5}$

Solution: The sample space consists of $99 - 10 + 1 = 90$ elements. The enumeration of those elements that contain a 7 are:

$$\{17,\ 27,\ 37,\ 47,\ 57,\ 67,\ 70,\ 71,\ 72,\ 73,\ 74,\ 75,\ 76,\ 77,\ 78,\ 79,\ 87,\ 97\}.$$

This set has 18 elements, so the probability of success is 18/90.

14. Find all integers n such that

$$\frac{5n+23}{n-7}$$

is also an integer.

Answer: $n \in \{-51,\ -22,\ 5,\ 6,\ 8,\ 9,\ 36,\ 65\}$

Solution:

$$\frac{5n+23}{n-7} = \frac{5n-35+35+23}{n-7} = 5 + \frac{58}{n-7}$$

By inspection you'll find $n \in \{-51,\ -22,\ 5,\ 6,\ 8,\ 9,\ 36,\ 65\}$.

15. For what value(s) of x is the reciprocal of $x+1$ equal to $x-1$?

Answer: $\pm\sqrt{2}$

Solution:

$$\begin{aligned} x-1 &= \frac{1}{x+1} \\ x^2-1 &= 1 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

16. The cross-sectional area of a certain tree is a linear function of time. If the radius of this circular cross-section was 2 feet in 1915 and 3 feet in 1990, what was its radius in 2005?

Answer: $\sqrt{10}$

Solution: We have a linear function that contains these points:

$$(1915,\ 4\pi);\quad (1990,\ 9\pi);\quad \left(2005,\ \pi r^2\right).$$

Using slope, we have:

$$m = \frac{9\pi - 4\pi}{1990 - 1915} = \frac{\pi}{15} = \frac{\pi r^2 - 9\pi}{2005 - 1990} = \frac{\pi r^2 - 9\pi}{15}.$$

Now, solve for r.

$$\begin{aligned} \frac{\pi}{15} &= \frac{\pi r^2 - 9\pi}{15} \\ 1 &= r^2 - 9 \\ 10 &= r^2 \\ \pm\sqrt{10} &= r \end{aligned}$$

17. An escalator from a subway platform to the street moves upward at a constant rate. Jane requires 40 seconds to walk up while the escalator is moving and takes 40 steps. Joe requires 50 seconds to walk up while the escalator is moving and takes 20 steps. How many steps of the escalator are required to go from the subway to the street if the escalator is not moving?

Answer: 120

Solution: Let r represent the rate of the escalator. Equating distance, we have:

$$40r + 40 = 50r + 20 \quad \Rightarrow \quad r = 2.$$

So the total distance is

$$40 \cdot 2 + 40 = 50 \cdot 2 + 20 = 120.$$

18. The sum of eleven consecutive integers is 2002. What is the smallest of these integers?

Answer: 177

Solution:

$$\begin{aligned} x + (x+1) + (x+2) + \cdots + (x+10) &= 2002 \\ 11x + 55 &= 2002 \\ 11x &= 1947 \\ x &= 177 \end{aligned}$$

19. If 1/3 and 1/4 are the lengths of the two legs of a right triangle, find the length of the hypotenuse.

Answer: 5/12

Solution: Solve for h.

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 = h^2 \quad \Rightarrow \quad \frac{1}{9} + \frac{1}{16} = h^2 \quad \Rightarrow \quad \frac{25}{9 \cdot 16} = h^2 \quad \Rightarrow \quad \pm\frac{5}{12} = h$$

20. Let f be a real-valued function such that

$$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$$

for all $x > 0$. Find $f(2)$.

Answer: 2000

Solution: Make two substitutions, $x = 1001$ and $x = 2$.

$$\begin{aligned} f(1001) + 2f(2) &= 3003 \\ f(2) + 2f(1001) &= 6 \end{aligned}$$

Now use elimination by multiplying the first equation by -2 to solve for $f(2)$.

$$\begin{cases} -2f(1001) - 4f(2) = -6006 \\ 2f(1001) + f(2) = 6 \end{cases}$$

When added, the result is:

$$-3f(2) = -6000 \quad \Rightarrow \quad f(2) = 2000.$$