2008 A. Shloming Mathematics Prize Examination
Essex County College—Division of Mathematics and Physics
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Name: ______________________________

Signature: ______________________________

If the question has choices, select one answer; if the question is open ended, write your final answer on the line provided. **Five points** for each correct answer. No calculators are allowed, and the use of cellular phones is strictly forbidden.

1. In the 1973 *Belmont Stakes*, Secretariat covered 12 furlongs[2] in 2 minutes, 24 seconds. What speed is that in miles per hour?

   Answer: 37.5

   Solution: We need to convert

$$\frac{12 \text{ furlongs}}{2 \text{ minutes } 24 \text{ seconds}} = \frac{12 \text{ furlongs}}{144 \text{ seconds}} = \frac{1 \text{ furlong}}{12 \text{ seconds}},$$

   into miles per hour by using unit conversion factors.

$$\frac{1 \text{ furlong}}{12 \text{ seconds}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{8 \text{ furlongs}} = \frac{75 \text{ miles}}{2 \text{ hours}}$$

2. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is

   (a) $\sqrt{2}+\sqrt{3}$

   (b) $\dfrac{7+3\sqrt{5}}{2}$

   (c) $1+2\sqrt{3}$

   (d) 3

   (e) $3+2\sqrt{2}$

[1] This document was prepared by Ron Bannon using $\LaTeX\,2_\varepsilon$ and was based on a variety of sources.
[2] 8 furlongs = 1 mile.

Answer: e

Solution:

$$\begin{aligned}\left(\sqrt{1+\sqrt{1+\sqrt{1}}}\right)^4 &= \left(\sqrt{1+\sqrt{2}}\right)^4 \\ &= \left(1+\sqrt{2}\right)^2 \\ &= 1+2\sqrt{2}+2\end{aligned}$$

3. Multiplying 4 times 2178 gives an answer that is the reverse of 2178; $4 \times 2178 = 8712$. Find the four digit number that when multiplied by 9 does the same thing.

Answer: 1089

Solution: The four digit number must start with 10, otherwise when we multiply by 9 we would exceed a four digit number. Furthermore the first digit must be 9 because when we multiply by 9 we need to get a 1. So we only need to determine the digit in the tens place. Our number is of the form $10x9$ and when multiplied by 9 is of the form $9x01$.

$$\begin{aligned}9\left(1000+10x+9\right) &= 9000+100x+1 \\ 9000+90x+81 &= 9000+100x+1 \\ 80 &= 10x \\ 8 &= x\end{aligned}$$

You may want to verify that

$$9 \cdot 1089 = 9801,$$

is in fact true.

4. $\cot(10) + \tan(5)$ is

   (a) $\csc(5)$

   (b) $\csc(10)$

   (c) $\sec(5)$

   (d) $\sec(10)$

   (e) $\sin(15)$

Answer: b

Solution:

$$\begin{aligned}
\cot(10) + \tan(5) &= \frac{\cos 10}{\sin 10} + \frac{\sin 5}{\cos 5} \\
&= \frac{\cos(2\cdot 5)}{\sin(2\cdot 5)} + \frac{\sin 5}{\cos 5} \\
&= \frac{\cos^2 5 - \sin^2 5}{2\sin 5\cos 5} + \frac{\sin 5}{\cos 5} \\
&= \frac{\cos^2 5 - \sin^2 5 + 2\sin^2 5}{2\sin 5\cos 5} \\
&= \frac{\cos^2 5 + \sin^2 5}{2\sin 5\cos 5} \\
&= \frac{1}{\sin 10}
\end{aligned}$$

5. One invention saves 30 percent on fuel; a second saves 45 percent; a third, 25 percent. If you use all three inventions at once, how much can you save?

Answer: $100\% \cdot \frac{569}{800} = \frac{569}{8}\% = 71\frac{1}{8}\% = 71.125\%$

Solution:

$$\begin{aligned}
(1-0.30)(1-0.45)(1-0.25) &= 1-x \\
\frac{7}{10}\cdot\frac{11}{20}\cdot\frac{3}{4} &= 1-x \\
\frac{231}{800} &= 1-x \\
x &= \frac{800}{800} - \frac{231}{800} \\
x &= \frac{569}{800}
\end{aligned}$$

6. The sum of the squares of the roots of $x^2 + 2hx = 3$ is 10. The absolute value of h is

   (a) 1

   (b) $\frac{1}{2}$

   (c) $\frac{3}{2}$

   (d) 2

   (e) none of these

Answer: a

Solution: The roots are:

$$x^2 + 2hx = 3 \quad \Rightarrow \quad x = -h \pm \sqrt{h^2+3}.$$

Here, solving for h.

$$\begin{aligned}
10 &= \left(-h - \sqrt{h^2+3}\right)^2 + \left(-h + \sqrt{h^2+3}\right)^2 \\
10 &= \left(2h^2 + 2h\sqrt{h^2+3} + 3\right) + \left(2h^2 - 2h\sqrt{h^2+3} + 3\right) \\
10 &= 4h^2 + 6 \\
4 &= 4h^2 \\
\pm 1 &= h
\end{aligned}$$

7. If $x, y > 0$, $\log_y x + \log_x y = \dfrac{10}{3}$ and $xy = 144$, then $\dfrac{x+y}{2}$ is

(a) $12\sqrt{2}$

(b) $13\sqrt{3}$

(c) 24

(d) 30

(e) 36

Answer: b

Solution:

$$\begin{aligned}
\log_y x + \log_x y &= \frac{10}{3} \\
\frac{\log_x x}{\log_x y} + \log_x y &= \frac{10}{3} \\
\frac{1}{\log_x y} + \log_x y &= \frac{10}{3} \\
3 + 3\left(\log_x y\right)^2 &= 10\log_x y \\
3\left(\log_x y\right)^2 - 10\log_x y + 3 &= 0 \\
\left(3\log_x y - 1\right)\left(\log_x y - 3\right) &= 0
\end{aligned}$$

Two solutions:

$$\log_x y = \frac{1}{3} \quad \text{or} \quad \log_x y = 3.$$

Or rewritten as:

$$y = \sqrt[3]{x} \quad \text{or} \quad y = x^3.$$

We know that:[3]

$$xy = 144 \quad \Rightarrow \quad xx^3 = 144 \quad \Rightarrow \quad x = 2\sqrt{3},$$

and[4]

$$xy = 144 \quad \Rightarrow \quad 2\sqrt{3}y = 144 \quad \Rightarrow \quad y = 24\sqrt{3}.$$

So, finally, we have:

$$\frac{x+y}{2} = \frac{2\sqrt{3}+24\sqrt{3}}{2} = \frac{26\sqrt{3}}{2} = 13\sqrt{3}.$$

8. Solve for x:

$$\sqrt{1+\sqrt{2+\sqrt{x}}} = 3.$$

Answer: $62^2 = 3844$

Solution: Lots of squaring to do!

$$\begin{aligned} \sqrt{1+\sqrt{2+\sqrt{x}}} &= 3 \\ 1+\sqrt{2+\sqrt{x}} &= 9 \\ \sqrt{2+\sqrt{x}} &= 8 \\ 2+\sqrt{x} &= 64 \\ \sqrt{x} &= 62 \\ x &= 62^2 = 3844 \end{aligned}$$

9. If x and y are positive integers for which $2^x 3^y = 1296$, what is the value of $x+y$?

Answer: 8

Solution:

$$\begin{aligned} 2^x 3^y &= 1296 \\ 2^x 3^y &= 2^4 3^4 \end{aligned}$$

Clearly $x + y = 8$

10. Given that they are made of the same material, which is heavier:

[3] $x > 0$
[4] $y > 0$

(a) a ball with a radius of 10 inches

(b) or 10 balls with a radius of 1 inch?

Answer: a

Solution:

$$V_a = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$$
$$V_b = 10 \cdot \frac{4}{3}\pi(1)^3 = \frac{40\pi}{3}$$

Clearly $V_a > V_b$.

11. The admission fee to a mathematics exhibition was \$15. When the fee was reduced, the (nonzero) number of customers per day went up by 50 percent and the amount of money collected per day went up by 25 percent. What was the reduced fee?

Answer: \$12.50

Solution: Let x represent the number of customers that pay the \$15 admission fee, and let r represent the reduced admission fee. We have

$$\begin{aligned} x \cdot 15 \cdot 1.25 &= x \cdot 1.5 \cdot r \\ 15 \cdot 1.25 &= 1.5 \cdot r \\ \frac{15 \cdot 1.25}{1.5} &= r \\ 12.5 &= r \end{aligned}$$

12. Given that $-4 \leq x \leq -2$ and $2 \leq y \leq 4$, what is the largest possible value of

$$\frac{x+y}{x}?$$

Answer: $\frac{1}{2}$

Solution:

$$\frac{x+y}{x} = 1 + \frac{y}{x}$$

By inspection, let $x = -4$ and $y = 2$.

13. What is the smallest positive integer n such that

$$\frac{16!}{n}$$

is a perfect square?

Answer: 1430

Solution:

$$\begin{aligned}\frac{16!}{n} &= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13\cdot14\cdot15\cdot16}{n}\\ &= \frac{2^{14}\cdot3^6\cdot5^2\cdot7^2\cdot2\cdot5\cdot11\cdot13}{n}\end{aligned}$$

So $n = 2\cdot5\cdot11\cdot13 = 1430$.

14. Jaisi runs twice as fast as she walks. When going to school one day she walks for twice the time she runs and takes 20 minutes to complete the trip. On her way home after school, she runs for twice the time that she walks. How long will it take Jaisi to get home after school?

Answer: 16 minutes.

Solution: Let r represent the walking rate and $2r$ the running rate. Let t represent the time running to school and $2t$ represent the time walking to school. The time to school is

$$2t + t = 20 \quad \Rightarrow \quad t = \frac{20}{3}.$$

The distance to and from school is:

$$r\cdot\frac{40}{3} + 2r\cdot\frac{20}{3} = \frac{80r}{3}$$

Let x represent the time spent walking home and $2x$ represent the time spent running home. The equation returning home is:

$$\begin{aligned}\frac{80r}{3} &= 2r\,(2x) + rx\\ \frac{80}{3} &= 4x + x\\ \frac{80}{3} &= 5x\\ \frac{16}{3} &= x.\end{aligned}$$

The total time going home is:

$$x + 2x = 3x = 3\cdot\frac{16}{3} = 16.$$

15. A square with area of 16 is partitioned into four congruent smaller squares. What is the area of the circle that passes through the centers of the four smaller squares?

Answer: 2π square units.

Solution: Each of the smaller squares has a diagonal that measures $2\sqrt{2}$, so the radius of the circle is $\sqrt{2}$, given an area 2π.

16. If prices go down by 20 percent, by what percent does your purchasing power increase?[5]

Answer: 25

Solution: Let x be the increase in purchasing power, we have

$$(1-0.2)(1+x)=1.$$

Solving this equation for x yields $x=0.25$.

17. A girl goes up a ski lift at 4 mph and comes down the ski slope at 24 mph. If the ski slope is the same length as the ski lift, and we ignore any time spent at the top, what is her average speed for the round trip in mph?

Answer: $\frac{48}{7}=6\frac{6}{7}$

Solution: Let t_u represent the time up and Let t_d represent the time down. We also know that $4t_u=24t_d$, so $t_u=6t_d$. The average speed is:

$$s_{\text{avg}}=\frac{4t_u+24t_d}{t_u+t_d}=\frac{4\cdot 6t_d+24t_d}{6t_d+t_d}=\frac{48}{7}$$

18. Put the following numbers in order from the least to the greatest: 1^{400}, 2^{300}, 3^{200}, 4^{100}.

Answer: $1^{400},\ 4^{100},\ 2^{300},\ 3^{200}$

Solution: Take the one-hundredth root of each term, 1^4, 2^3, 3^2, 4^1, and then order these.

19. The roots of x^2+ax+b are the squares of the roots of x^2+x+1. Compute the values of a and b, and write the answer as the ordered pair $(a,\ b)$.

Answer: $(1,\ 1)$

Solution: The roots of x^2+x+1 are:

$$\frac{-1\pm i\sqrt{3}}{2}.$$

Squaring these roots yields:

$$\left(\frac{-1\pm i\sqrt{3}}{2}\right)^2=\frac{-1\pm i\sqrt{3}}{2}.$$

So $a=b=1$.

[5] Purchasing power means the amount of goods you can purchase for a fixed amount of money.

20. Let

$$f(x) = \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}.$$

What is the simplest equivalent form of $f(x)$?

Answer: $\cos 2x$

Solution:

$$\begin{aligned} f(x) &= \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x} \\ &= \sqrt{\sin^4 x + 4\left(1 - \sin^2 x\right)} - \sqrt{\cos^4 x + 4\left(1 - \cos^2 x\right)} \\ &= \sqrt{\sin^4 x - 4\sin^2 x + 4} - \sqrt{\cos^4 x - 4\cos^2 x + 4} \\ &= \sqrt{\left(\sin^2 x - 2\right)^2} - \sqrt{\left(\cos^2 x - 2\right)^2} \\ &= \left|\sin^2 x - 2\right| - \left|\cos^2 x - 2\right| \\ &= 2 - \sin^2 x - \left(2 - \cos^2 x\right) \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \end{aligned}$$