Essex County College—Division of Mathematics
2009 A. Shloming Mathematics Prize Examination

This Prize Examination has 20 questions, for a total of 100 points.

Last Name: ____

First Name: ____

Phone or email: ____

Prize Examination Honor Code: The Prize Examination Honor Code is a statement on academic integrity, it articulates reasonable expectations of students and teachers in establishing and maintaining the highest standards in academic work:

1. that they will not give or receive aid in taking this Prize Examination, including the use of notes and electronic devises;
2. that they will not use any communication device while taking this Prize Examination, either in the room or while on a break. If you have a device that rings or vibrates during the contest, *DO NOT ANSWER IT* or look at it. Prior to the Prize Examination you must turn these devices off and store them away from you for the duration of the Prize Examination. Your Prize Examination will be invalidated and no score may be earned if you use any such device while in the Prize Examination room;
3. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Prize Examination Honor Code;
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Please sign your name below to record that you have reviewed this Prize Examination Honor Code and will abide by these expectations at all times during this Prize Examination.

Signature: ____

If the question has choices, select one answer; if the question is open ended, write your final answer on the line provided. You do not need to show your work and you will not be given partial credit. *Five points* for each correct answer, and there's no penalty for incorrect answers. No calculators are allowed, and the use of cellular phones is strictly forbidden.

1. 5 points Find the product of all real values of x that satisfy the equation

$$6\,|x-6| = |x+6|\,.$$

1. 36

Solution: There are two solutions as follows.

$$x+6 = 6\,(x-6) \quad \Rightarrow \quad x = \frac{42}{5},$$

and

$$x+6 = 6\,(6-x) \quad \Rightarrow \quad x = \frac{30}{7}.$$

The product of these two solutions is

$$\frac{42}{5} \cdot \frac{30}{7} = 36.$$

Although not required, here's the graph.

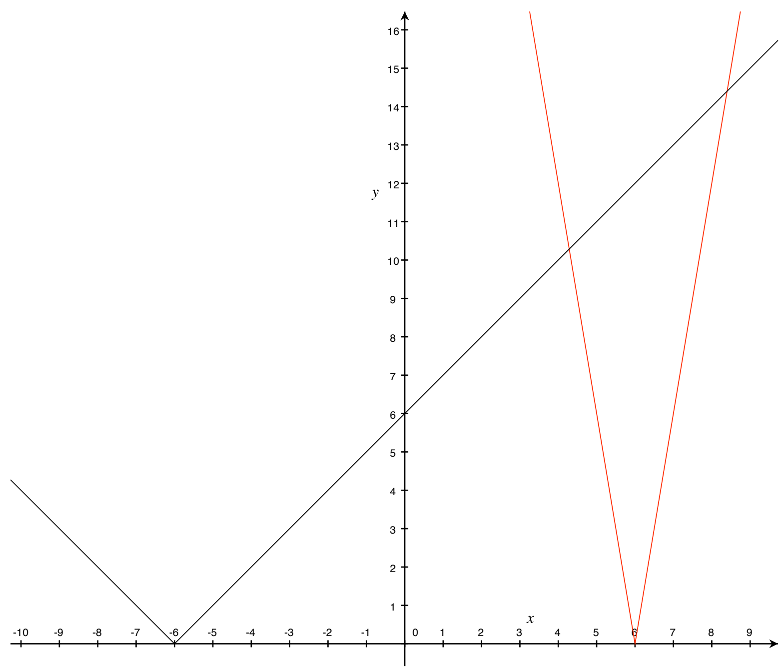

Figure 1: Partial graph of $y = 6\,|x-6|$ [Red] and $y = |x+6|$.

2. 5 points You are given that $xy = 6$ and $x^2y + xy^2 + x + y = 63$. Calculate the value of $x^2 + y^2$.

2. 69

Solution: Using $xy = 6$, we have:

$$\begin{aligned} x^2y + xy^2 + x + y &= 63 \\ x(xy) + y(xy) + x + y &= 63 \\ 6x + 6y + x + y &= 63 \\ 7x + 7y &= 63 \\ x + y &= 9 \\ x^2 + y^2 + 2xy &= 81 \\ x^2 + y^2 + 12 &= 81 \\ x^2 + y^2 &= 69 \end{aligned}$$

3. [5 points] If one root of $x^3 - 5x^2 + 5x - 1 = 0$ is $2 - \sqrt{3}$, then the sum of the other two roots is

A. $-7 + \sqrt{3}$ B. $-1 + \sqrt{3}$ **C.** $3 + \sqrt{3}$ D. $-3 + \sqrt{3}$ E. 5 F. Not listed.

3. **C**

Solution: If $2 - \sqrt{3}$ is a root, then we have its conjugate $2 + \sqrt{3}$ as another root. Furthermore, we should have one rational root left, and using the **Rational Root Theorem**, where we only have two choices, either 1 or -1 and can easily verify that the solution is 1. So we have the two missing roots and their sum is:

$$1 + 2 + \sqrt{3} = 3 + \sqrt{3}.$$

4. [5 points] Find the integers t and the digit symbolized by a in the following:

$$[3(230 + t)]^2 = 492a04$$

4. $a = 8$**,** $t = 4$ **or** $t = -464$

Solution: $492a04$ must be divisible by 9, so we have

$$\frac{4 + 9 + 2 + a + 0 + 4}{9} = \frac{19 + a}{9} \quad \Rightarrow \quad a = 8$$

Now we have an equation in t.

$$\begin{aligned} [3(230 + t)]^2 &= 492804 \\ (230 + t)^2 &= 54756 \\ 230 + t &= \pm 234 \\ t &= -230 \pm 234 \end{aligned}$$

5. 5 points Two clocks both show eight o'clock. One clock gains one minute each hour, whereas the other clock loses one minute each hour. In how many hours will both clocks again show the identical time?

5. 360

Solution: Each hour the two clocks will be out of sync by two minutes. In 30 hours they will be 60 minutes (one hour) apart. When they are exactly 12 hours apart they will be showing the same time, so we have $12 \times 30 = 360$ hours.

6. 5 points The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by:

A. $\log 3$ B. $\frac{1}{2}\log 6$ C. $1 + \log\left(\frac{3}{2}\right)$ **D.** $1 + \frac{\log 3}{\log 2}$ E. $\frac{\log 2}{\log 3}$ F. Not listed.

6. **D**

Solution: First factor.

$$\begin{aligned} 2^{2x} - 8 \cdot 2^x + 12 &= 0 \\ \left(2^x - 2\right)\left(2^x - 6\right) &= 0 \end{aligned}$$

Using the **Zero Product Rule** we have,

$$2^x - 2 = 0 \quad \Rightarrow \quad x = 1,$$

and

$$2^x - 6 = 0 \quad \Rightarrow \quad x = \frac{\log 6}{\log 2} = \frac{\log 2 + \log 3}{\log 2} = 1 + \frac{\log 3}{\log 2}.$$

7. 5 points Find the sum of the roots of $\tan^2 x - 9\tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ **D.** 3π E. 4π

7. **D**

Solution:

$$\begin{aligned} \tan^2 x - 9\tan x + 1 &= 0 \\ \sin^2 x - 9\sin x\cos x + \cos^2 x &= 0 \\ 1 - 9\sin x\cos x &= 0 \\ 1 &= 9\sin x\cos x \\ \frac{2}{9} &= 2\sin x\cos x \\ \frac{2}{9} &= \sin 2x \end{aligned}$$

The solution to this equation that are between $x = 0$ and $x = 2\pi$ radians, is:

$$\frac{1}{2}\arcsin\left(\frac{2}{9}\right);\ \frac{\pi}{2}-\frac{1}{2}\arcsin\left(\frac{2}{9}\right);\ \pi+\frac{1}{2}\arcsin\left(\frac{2}{9}\right);\ \frac{3\pi}{2}-\frac{1}{2}\arcsin\left(\frac{2}{9}\right).$$

The sum is 3π.

8. 5 points Bobbi and Ellie each chose a secret five-digit number. Ellie's number was smaller than Bobbi's. The sum of the numbers was 50581, and the difference was 12965. When each entered her number in a calculator and looked at it upside down, what did Bobbi see?

8. **ELLIE**

Solution: Let e be Ellie's number and b be Bobbi's number. We have

$$b > e, \quad b + e = 50581, \quad b - e = 12965.$$

Solving this system gives $b = 31773$ and $e = 18808$. Upside down, Bobbi's number looks like "ELLIE" and Ellie's number looks like "BOBBI".

9. 5 points If $2f(x) + f(1-x) = x^2$ for all x, then $f(x) =$?

9. $\left(x^2+2x-1\right)/3$

Solution: Make a substitution, use $1 - x$:

$$\begin{aligned} 2f(1-x) + f(x) &= (1-x)^2 \\ 2f(1-x) + f(x) &= x^2 - 2x + 1 \end{aligned}$$

Now you have two equations where $f(1-x)$ can be eliminated.

$$\begin{cases} f(1-x) + 2f(x) = x^2 \\ 2f(1-x) + f(x) = x^2 - 2x + 1 \end{cases}$$

Multiply the top equation by -2.

$$\begin{cases} -2f(1-x) - 4f(x) = -2x^2 \\ 2f(1-x) + f(x) = x^2 - 2x + 1 \end{cases}$$

Add these two equations together.

$$-3f(x) = -x^2 - 2x + 1$$

Solve for $f(x)$.

$$f(x) = \frac{x^2 + 2x - 1}{3}$$

10. [5 points] The mathematician Augustus De Morgan lived his entire life during the 1800s. In the last year of his life he announced, "Once I was x years old in the year x^2." In what year was he born?

10. 1806

Solution: Some trial and error, but note that

$$1800 \leq x^2 \leq 1899,$$

has only one integer solution, that is $x = 43$. So Augustus De Morgan was 43 years old in 1849, therefore he was born in 1806.

11. [5 points] A function f is defined by $f(z) = i\bar{z}$, where $i = \sqrt{-1}$ and $\bar{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

11. 2

Solution: Let $z = a + bi$ and $\bar{z} = a - bi$. We're given:

$$\begin{aligned} f(z) &= z \\ f(a+bi) &= a+bi \\ i(a-bi) &= a+bi \\ ai - bi^2 &= a+bi \\ b + ai &= a+bi. \end{aligned}$$

From this we can conclude that $a = b$, since $|z| = 5$ we have:

$$\begin{aligned} |z| &= 5 \\ a^2 + b^2 &= 25 \\ 2a^2 &= 25. \end{aligned}$$

Clearly we have two solutions.

12. [5 points] In a sequence, every term after the second term is twice the sum of the two preceeding terms. The seventh term of the sequence is 8, and the ninth term is 24. What is the eleventh term of the sequence?

12. 160

Solution: Here we have $a_7 = 8$ and $a_9 = 24$, to find a_8:

$$\begin{aligned} a_9 &= 2(a_8 + a_7) \\ 24 &= 2(a_8 + 8) \\ 12 &= a_8 + 8 \\ 4 &= a_8. \end{aligned}$$

To find a_{11}:

$$\begin{aligned} a_{10} &= 2(a_9 + a_8) \\ a_{10} &= 2(24 + 4) \\ a_{10} &= 56 \\ a_{11} &= 2(a_{10} + a_9) \\ a_{11} &= 2(56 + 24) \\ a_{11} &= 160. \end{aligned}$$

13. 5 points If $t_1 = 1$, $t_2 = -1$, and

$$t_n = \left(\frac{n-3}{n-1}\right) \cdot t_{n-2}$$

for $n \geq 3$, find the value of t_{1998}.

13. $-1/1997$

Solution:

$$\begin{aligned} t_3 &= \left(\frac{3-3}{3-1}\right) \cdot t_{3-2} = 0 \\ t_4 &= \left(\frac{4-3}{4-1}\right) \cdot t_{4-2} = \frac{1}{3} \cdot t_2 \\ t_5 &= \left(\frac{5-3}{5-1}\right) \cdot t_{5-2} = \frac{2}{4} \cdot t_3 \\ t_6 &= \left(\frac{6-3}{6-1}\right) \cdot t_{6-2} = \frac{3}{5} \cdot t_4 = \frac{3}{5} \cdot \frac{1}{3} \cdot t_2 \end{aligned}$$

Inspecting for patterns we have:

$$t_{1998} = \frac{1995}{1997} \cdot \frac{1993}{1995} \cdot \frac{1991}{1993} \cdot \cdots \cdot \frac{5}{7} \cdot \frac{3}{5} \cdot \frac{1}{3} \cdot t_2$$

14. 5 points A $5 \times 5 \times 5$ cube with all its faces painted red is cut into 125 cubes that are $1 \times 1 \times 1$. Some of these new cubes will have red faces. The cubes are placed into a bag and throughly mixed. One is drawn at random and rolled like a die. What is the probability that a red face shows assuming that all cubes are fair die?

14. $1/5$

Solution: You have 125 cubes and $125 \cdot 6$ faces, of these faces, $25 \cdot 6$ will be red. So we have a probability of success

$$\frac{25 \cdot 6}{125 \cdot 6} = \frac{1}{5}.$$

15. 5 points What geometric sequence can be represented by x, $3x+1$, and $6x+2$?

15. $\{-1,\ -2,\ -4\}$

Solution: Using common ratios, we have:

$$\begin{aligned} \frac{3x+1}{x} &= \frac{6x+2}{3x+1} \\ 9x^2+6x+1 &= 6x^2+2x \\ 3x^2+4x+1 &= 0 \\ (3x+1)(x+1) &= 0, \end{aligned}$$

with solution $x=-1$ or $x=-1/3$, but x cannot be $-1/3$. So we have $\{-1,\ -2,\ -4\}$.

16. 5 points Find the sum of the series

$$1\,(1!) + 2\,(2!) + 3\,(3!) + \cdots + n\,(n!)\,.$$

16. $(n+1)! - 1$

Solution: This is a tough one.

$$\begin{aligned} 1\,(1!) + 2\,(2!) + 3\,(3!) + \cdots + n\,(n!) &= 2\,(1!) + 3\,(2!) + 4\,(3!) + \cdots + (n+1)\,(n!) \\ &\quad - 1! - 2! - 3! - \cdots - n! \\ &= 2! + 3! + 4! + \cdots + (n+1)! \\ &\quad -1! - 2! - 3! - \cdots - n! \\ &= (n+1)! - 1 \end{aligned}$$

17. 5 points For any three real numbers a, b, c with $b \neq c$, the operation

$$\P\,(a,\ b,\ c) = \frac{a}{b-c},$$

what is

$$\P\,(\P\,(1,\ 2,\ 3)\,,\ \P\,(2,\ 3,\ 1)\,,\ \P\,(3,\ 1,\ 2))$$

17. $-1/4$

Solution:
$$\P\left(\P(1,\ 2,\ 3),\ \P(2,\ 3,\ 1),\ \P(3,\ 1,\ 2)\right) = \P(-1,\ 1,\ -3) = -\frac{1}{4}$$

18. 5 points What is the number of solutions $(x,\ y)$ of the equation $3x + y = 100$, where x and y are positive integers?

18. 33

Solution: Solve for y.
$$y = 100 - 3x$$
For $y > 0$ we have a range for x from 1 through 33.

19. 5 points How many different real roots does the polynomial equation have?
$$x^{11} + x^{10} + x^9 + \cdots + 1 = 0$$

19. 1

Solution: Long division may be helpful.
$$\begin{aligned} x^{11} + x^{10} + x^9 + \cdots + 1 &= 0 \\ (x+1)\left(x^{10} + x^8 + x^6 + x^4 + x^2 + 1\right) &= 0 \end{aligned}$$
There is no real solution to the equation $x^{10} + x^8 + x^6 + x^4 + x^2 + 1 = 0$, and $x + 1 = 0$ has one real solution.

20. 5 points If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 1$ exactly once, then m^2 is

A. $\frac{1}{2}$ B. $\frac{2}{3}$ **C.** $\frac{3}{4}$ D. $\frac{4}{5}$ E. $\frac{5}{6}$ F. Not listed.

20. **C**

Solution: You'll need to find the slope of the tangent by taking the derivative of $x^2 + 4y^2 = 1$ implicitly.
$$y' = -\frac{x}{4y}$$
The slope for the line is computed by taking two points on the line, the y-intercept $(0,\ 1)$ and a point on the ellipse $(x,\ y)$. Here's the slope equation:
$$-\frac{x}{4y} = \frac{y-1}{x-0} \quad \Rightarrow \quad 4y = 4y^2 + x^2.$$

Since x and y in this equation must be on the ellipse, we have:

$$4y = 4y^2 + x^2 \quad \Rightarrow \quad y = \frac{1}{4} \quad \text{and} \quad x = \pm\frac{\sqrt{3}}{2}.$$

Again, the slope is:

$$y' = -\frac{x}{4y} = \pm\frac{\sqrt{3}}{2}.$$

Although not required, a graph may be helpful.

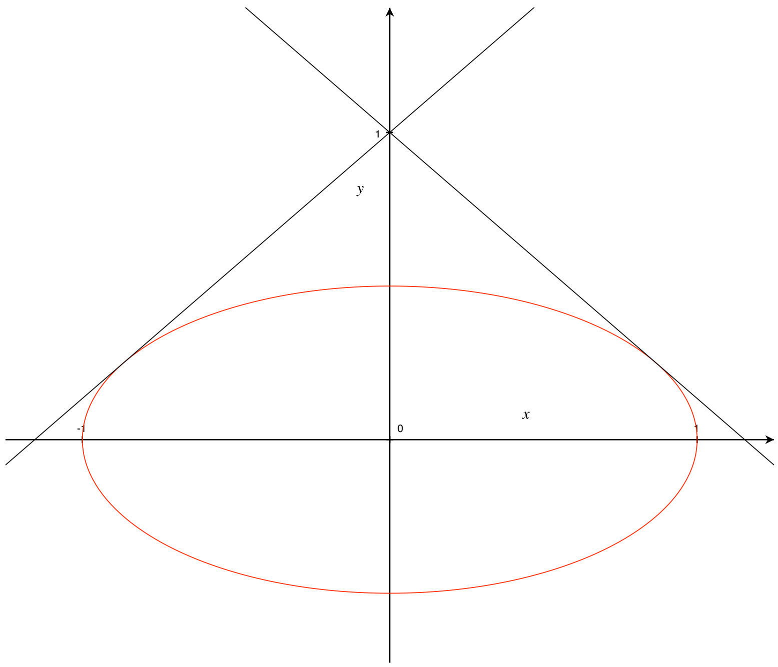

Figure 2: Partial graph of $x^2 + 4y^2 = 1$ [Red] and $y = mx + 1$, $m = \pm\sqrt{3}/2$.

END OF EXAM

In Honor of Adolph Shloming

In 1903, Adolph Shloming was born in a small village to a poor family in Romania. His education took place in Bucharest where he studied languages in addition to mathematics. As a young man, Adolph spoke a half dozen languages including German, Russian, Hungarian, Hebrew, Yiddish and of course Romanian. Along with his love of languages was his passion for chess and mathematics. His long hours of concentration over a chessboard prepared him to be analytical and patient. Figuring out the optimum move involved examining many possibilities and their consequences. These characteristics helped him solve a variety of mathematical problems; many of them recreational. To Adolph, problem solving was as enjoyable and challenging as an exciting game of chess!

To his credit, he was a master chess player as well as a gifted problem solver.

In 1930, he left Romania for the United States and lived in Harlem. While in New York City, he was active in the union movement. His main concern was the improvement of the working conditions in the sweatshops, which were quite numerous in the 30's. In addition, he was a member of the Workmen's Circle which was a social and educational association for workers who recently emigrated from Europe. The Workmen's Circle had many educational programs for newly arrived families. Adolph supported the idea that the language for the parents to master was English and that they should speak it at home. The second language for the children was the language of mathematics. He held to the idea that mathematics was the gateway to success.

In keeping with Adolph's love of mathematics, the mathematics department at Essex County College offers the A. Shloming Mathematics Prize Exam that is open to all students at the college. This annual college level mathematics competition has been held for more than a decade. The participants are challenged with problems from algebra, geometry, trigonometry and elementary calculus. Being exposed to challenging and non-routine problems results in student enthusiasm for mathematics.

Adolph's son, Robert Shloming, sponsors this prize exam in honor of his father. Following his father's dedication to mathematics, Robert earned a Ph.D. from New York University and has taught in the Essex County College mathematics department for more than three decades. By offering this mathematics exam, Dr. Shloming hopes that the best and the brightest Essex County College students will share in the challenges and joys of this competition. But it is really more than a competition. One of the objectives is to create a community of students that will be enthusiastic and motivated to discuss mathematical problems and consider careers in mathematics or related fields.

I wish to thank the entire mathematics department for supporting this endeavor. Also I would like to extend my special gratitude to Prof. Ron Bannon. Without Ron's tireless efforts to develop, publicize, proofread and score the prize exam, Essex County College would not have this wonderful event. Last, but certainly not least, I want to congratulate all the participants who have taken a courageous step in experiencing a mathematics competition.

Good Luck to All, Dr. Robert Shloming