Essex County College—Division of Mathematics
$2 \cdot 3 \cdot 5 \cdot 67 = 2010$ A. Shloming Mathematics Prize Examination

This Prize Examination has 20 questions, for a total of 100 points.

Last Name: ______________________________

First Name: ______________________________

Phone or email: ______________________________

Prize Examination Honor Code: The Prize Examination Honor Code is a statement on academic integrity, it articulates reasonable expectations of students and teachers in establishing and maintaining the highest standards in academic work:

1. that they will not give or receive aid in taking this Prize Examination, including the use of notes and electronic devises;
2. that they will not use any communication device while taking this Prize Examination, either in the room or while on a break. If you have a device that rings or vibrates during the contest, *DO NOT ANSWER IT* or look at it. Prior to the Prize Examination you must turn these devices off and store them away from you for the duration of the Prize Examination. Your Prize Examination will be invalidated and no score may be earned if you use any such device while in the Prize Examination room;
3. that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Prize Examination Honor Code;
4. that they will only turn in their Prize Examination if they are able to honestly state "I do hereby affirm, at the close of this Prize Examination, that I had no unlawful knowledge of the questions or answers prior to the contest and that I have neither given nor received assistance in answering any of the questions during this Prize Examination."

Please sign your name below to record that you have reviewed this Prize Examination Honor Code and will abide by these expectations at all times during this Prize Examination.

Signature: ______________________________

If the question has choices, select one answer; if the question is open ended, write your final answer on the line provided. You do not need to show your work and you will not be given partial credit. *Five points* for each correct answer, and there's no penalty for incorrect answers. No calculators are allowed, and the use of cellular phones is strictly forbidden.

1. 5 points On Monday a store put out 100 watermelons to be sold, and some were sold. On Tuesday the number left over was doubled, and sales were double what they were on Monday. On Wednesday, the number left over was tripled, and sales were triple what they were on Monday, leaving none left over. How many were sold each day?

Solution: Let x be the number sold each day. At the end of each day we have:

$$100 - x; \quad 2(100 - x) - 2x = 200 - 4x; \quad 3(200 - 4x) - 3x = 600 - 15x.$$

Clearly, since the last day has zero watermelons left, x must be 40.

Answer: 40, 80, 120

2. 5 points If $f(x) = \dfrac{x^4 + x^2}{x + 1}$, then $f(i)$, where $i = \sqrt{-1}$, is equal to

A. $i + 1$ B. 1 C. -1 D. 0 E. $-1 - i$

Solution:

$$f(i) = \frac{i^4 + i^2}{i + 1} = \frac{1 - 1}{i + 1} = 0$$

Answer: d

3. 5 points The Arzadun family is about to embark on an 18000-mile car trip through Europe. The tires on their car are all new, but each is good for only 12000 miles. What is the smallest number of new spare tires they should take along if they want to make the trip without having to buy any new tires along the way?

Solution: Each tires gives 12000 miles, but we need to go 18000 miles, or $4 \cdot 18000 = 72000$ tire miles. So we have

$$\frac{72000}{12000} = 6,$$

tires needed for the trip. Here's how, drive 6000 miles and change the two back tires for the two new spares. Store the two used tires as spares. Now drive another 6000 miles, and then exchange the two front tires for the two used spares.

Answer: 2

4. 5 points A circular ferris wheel has a radius of 8 meters and rotates at a rate of 12 degrees per second. At $t = 0$, a seat is at its lowest point, which is two meters above the ground. Determine how high above the ground the seat is at $t = 40$ seconds.

Solution: First you'll need to determine the rotation from the starting position.

$$\frac{12^\circ}{\text{sec}} \cdot 40 \text{ sec} = 480^\circ$$

We know that $480°$ of rotation from the starting position will result in $30°$ rise from the center, which means that it will be $8 + 8 \cdot \sin 30° = 12$ meters above the starting position, for a total of 14 meters.

Answer: 14 meters

5. 5 points If the points $(1, y_1)$ and $(-1, y_2)$ lie on the graph of $y = ax^2+bx+c$, and $y_1-y_2 = -6$, then b equals

A. -3 B. 0 C. 3 D. $\sqrt{ac}$ E. $\frac{a+c}{2}$

Solution: Substituting $(1, y_1)$ into $y = ax^2 + bx + c$, yields:

$$y_1 = a + b + c.$$

Substituting $(-1, y_2)$ into $y = ax^2 + bx + c$, yields:

$$y_2 = a - b + c.$$

Subtracting y_2 from y_1 yields:

$$y_1 - y_2 = 2b = -6 \quad \Rightarrow \quad b = -3.$$

Answer: a

6. 5 points A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is

A. $\frac{4}{\pi}$ B. $\frac{\pi}{\sqrt{2}}$ C. $\frac{4}{1}$ D. $\frac{\sqrt{2}}{\pi}$ E. $\frac{\pi}{4}$

Solution: The square's perimeter is $4x$ and the circle's perimeter is $2\pi r$, so, since they are equal we have:

$$4x = 2\pi r \quad \Rightarrow \quad x^2 = \frac{\pi^2 r^2}{4}.$$

The ratio of the area of the circle to the area of the square is:

$$\frac{\pi r^2}{x^2} = \frac{\pi r^2}{\pi^2 r^2/4} = \frac{4}{\pi}$$

Answer: a

7. 5 points A 100-pound watermelon is 95 percent water by weight. It is dehydrated until it is 90 percent water by weight. What is its weight after dehydration.

Solution: We start with 95 pounds water and 5 pounds of solids. So, after dehydration, the solids remain and are now 10 percent of the watermelon's weight. Let x be the watermelon's dehydrated weight.

$$\frac{5}{x} = 0.10 \quad \Rightarrow \quad x = 50$$

You could also do this using the water instead.

$$\frac{95 - x}{100 - x} = 0.90 \quad \Rightarrow \quad x = 50$$

Answer: 50 pounds

8. 5 points The diagonal of one square is four times the length of the diagonal of another. How many times larger is the area of the larger square?

Solution: Let the diagonal of the smaller square be $d/\sqrt{2}$, and the diagonal of the larger square be $4d/\sqrt{2}$. The area of the smaller square is $d^2/2$, and the area of the larger square is $8d^2$. Clearly, 16 times larger.

Answer: 16 times

9. 5 points Can the sum of two primes be prime?

Solution: $2 + 3 = 5$, nothing more to say.

Answer: Yes.

10. 5 points Today my daughter is 1/3 my age. Five years ago she was one 1/4 my age then. How old is my daughter now?

Solution: Let d be my daughter's age now, and m be my age now. We have,

$$d = \frac{m}{3} \quad \Rightarrow \quad 3d = m.$$

Five years ago we have,

$$d - 5 = \frac{m - 5}{4} \quad \Rightarrow \quad 4d - 20 = m - 5.$$

Now substitute $m = 3d$ into $4d - 20 = m - 5$ and solve for d.

$$\begin{aligned} 4d - 20 &= m - 5 \\ 4d - 20 &= 3d - 5 \\ d &= 15 \end{aligned}$$

Answer: 15

11. 5 points The volume of a cube is 8 times greater than the volume of another cube. What is the relationship of their surface areas?

Solution: Let x be the measure of edge of the larger cube and y be the measure of edge of the smaller cube. The volumes are:

$$V_x = x^3 \quad \text{and} \quad V_y = y^3.$$

The relationships between the volumes is:

$$x^3 = 8y^3 \quad \Rightarrow \quad x = 2y.$$

The surface areas are:

$$S_x = 6x^2 = 6\,(2y)^2 = 24y^2 \quad \text{and} \quad S_y = 6y^2.$$

So we have $S_x = 4S_y$.

Answer: 4 times greater.

12. 5 points Some women and some horses are in a stable. In all, there are 22 heads and 72 legs. How many women and how many horses are in the stable?

Solution: Let w represent the number of women and h the number of horses. We have

$$\begin{cases} w + h = 22 \\ 2w + 4h = 72 \end{cases},$$

a simple system of linear equations with solution $h = 14$ and $w = 8$.

Answer: 14 horses and 8 women.

13. 5 points The sum of 49 consecutive integers is $7^5 = 16807$. What is the median?

Solution: The integers are:

$$x,\ x+1,\ x+2,\ \cdots,\ x+47,\ x+48$$

There sum is:

$$\begin{aligned} 49x + 1 + 2 + 3 + \cdots 48 &= 7^5 \\ 49x + \frac{48\,(48+1)}{2} &= 7^5 \\ 49x + \frac{48 \cdot 49}{2} &= 7^5 \\ x + 24 &= 7^3 \\ x + 24 &= 343 \\ x &= 319 \end{aligned}$$

The median in the center number:

$$\underbrace{319,\ 320,\ 321,\ \cdots,\ 342,}_{\text{first 24}}\ \underbrace{343,}_{\text{median}}\ \underbrace{344,\ \cdots,\ 365,\ 366,\ 367.}_{\text{last 24}}$$

Answer: 343

14. 5 points Circle C_1 has a center $(0,\ 2)$ with radius 2, and circle C_2 has a center $(2,\ 0)$ with radius 2. The circles overlap in the first quadrant. What is the area of this overlap?

Solution: And then visualize a 2×2 square containing the region of interest. The area that we want is obtained by noting that the two equal sectors[1] completely fill the square, but overlap in the region that we want to measure. So

$$\pi + \pi - 4 = 2\pi - 4.$$

Answer: $2\pi - 4$ square units.

Graphing the circles is encouraged.

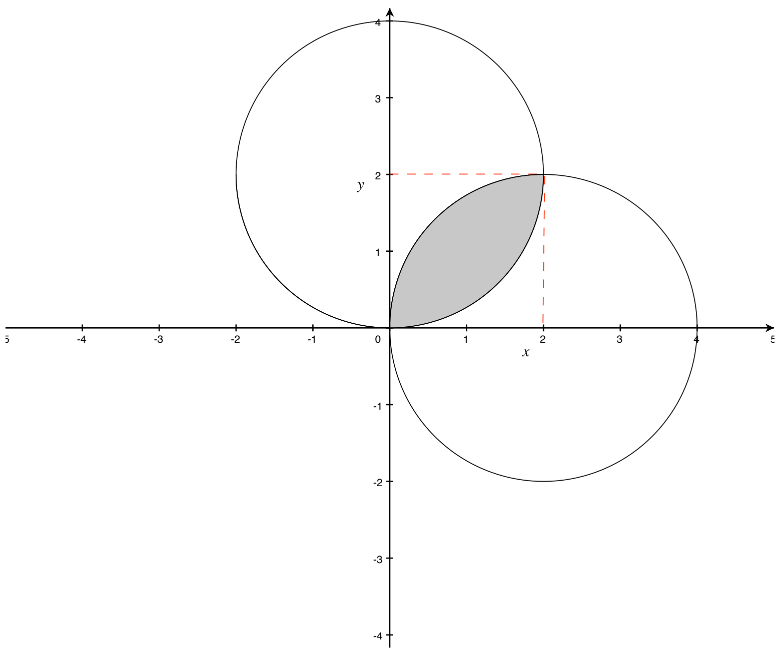

Figure 1: Circles C_1 and C_2

15. 5 points Two numbers are such that their difference, their sum, and their product are in the ratio $1 : 7 : 24$, respectively. What is their product?

Solution: We have $x - y : x + y : xy$, resulting in $x + y = 7(x - y)$ and $24(x + y) = 7xy$. Solving these equations gives $x = 8$ and $y = 6$; or $x = 0$ and $y = 0$. However these are ratios, hence $x = 8$ and $y = 6$ is the only valid solution.

Answer: 48

16. 5 points Before going to a garage sale, Elise counted the money she was taking along. After an hour, she counted it again and found that she had spent exactly half of it. The number of cents she now had equaled the number of dollars she started with; the number of dollars she now had was half the number of cents she started with. How much did she spend?

Solution: Let d represent the number of dollars started with and let c represent the number of cents started with. The starting amount of money is

$$100d + c,$$

and you now have half that amount

$$\frac{100d + c}{2}.$$

Here's the equation:

$$\frac{100d + c}{2} = 100\left(\frac{c}{2}\right) + d \quad \Rightarrow \quad 98d = 99c.$$

So you must have started with \$99.98.

Answer: \$49.99

17. 5 points The number of seconds in six weeks equals $n!$. Find n.

Solution:

$$6 \cdot 7 \cdot 24 \cdot 60 \cdot 60 = 6 \cdot 7 \cdot 2^3 \cdot 3 \cdot 2^2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

Answer: 10

18. 5 points For how many positive integers n is $n^3 - 8n^2 + 20n - 13$ prime?

Solution: Using the rational root theorem and long division to factor, we get:

$$(n - 1)\left(n^2 - 7n + 13\right).$$

Now, if this is prime, one of the factors must be 1, so set $n - 1 = 1$ and see what happens.

$$1 \cdot 3$$

So, for $n = 2$, $n^3 - 8n^2 + 20n - 13$ is prime.

Now, again if this is prime, the other factor should also be 1, so set $n^2 - 7n + 13 = 1$ and see what happens. Here, we need to solve

$$n^2 - 7n + 13 = 1 \quad \Rightarrow \quad (n-3)(n-4) = 0.$$

Now, we need to check $n = 3$ and $n = 4$ and let's see what happens.

$$2 \cdot 1 \quad \text{and} \quad 3 \cdot 1$$

Answer: 3

19. 5 points What is the slope of the tangent line to $y = x \sin x$ at $x = \pi$.

A. $-\pi$ B. π C. 0 D. 1 E. -1

Solution: Find the derivative.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x + x \cos x$$

Evaluate the derivative at $x = \pi$.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\pi} = \sin \pi + \pi \cos \pi = -\pi$$

Answer: a

20. 5 points What positive number x satisfy

$$\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3?$$

Solution: Cube both sides.

$$\begin{aligned}
(x+9) - 3(x+9)^{2/3}(x-9)^{1/3} + 3(x+9)^{1/3}(x-9)^{2/3} - (x-9) &= 27 \\
18 - 3(x+9)^{2/3}(x-9)^{1/3} + 3(x+9)^{1/3}(x-9)^{2/3} &= 27 \\
3(x+9)^{1/3}(x-9)^{2/3} - 3(x+9)^{2/3}(x-9)^{1/3} &= 9 \\
(x+9)^{1/3}(x-9)^{2/3} - (x+9)^{2/3}(x-9)^{1/3} &= 3 \\
(x+9)^{1/3}(x-9)^{1/3}\left((x-9)^{1/3} - (x+9)^{1/3}\right) &= 3 \\
\left(x^2-81\right)^{1/3}\left((x-9)^{1/3} - (x+9)^{1/3}\right) &= 3 \\
\left(x^2-81\right)^{1/3}\left(\sqrt[3]{x-9} - \sqrt[3]{x+9}\right) &= 3 \\
\left(x^2-81\right)^{1/3}(-3) &= 3 \\
\left(x^2-81\right)^{1/3} &= -1 \\
x^2 - 81 &= -1 \\
x^2 &= 80
\end{aligned}$$

Answer: $\sqrt{80}$

END OF EXAM

In Honor of Adolph Shloming

In 1903, Adolph Shloming was born in a small village to a poor family in Romania. His education took place in Bucharest where he studied languages in addition to mathematics. As a young man, Adolph spoke a half dozen languages including German, Russian, Hungarian, Hebrew, Yiddish and of course Romanian. Along with his love of languages was his passion for chess and mathematics. His long hours of concentration over a chessboard prepared him to be analytical and patient. Figuring out the optimum move involved examining many possibilities and their consequences. These characteristics helped him solve a variety of mathematical problems; many of them recreational. To Adolph, problem solving was as enjoyable and challenging as an exciting game of chess!

To his credit, he was a master chess player as well as a gifted problem solver.

In 1930, he left Romania for the United States and lived in Harlem. While in New York City, he was active in the union movement. His main concern was the improvement of the working conditions in the sweatshops, which were quite numerous in the 30's. In addition, he was a member of the Workmen's Circle which was a social and educational association for workers who recently emigrated from Europe. The Workmen's Circle had many educational programs for newly arrived families. Adolph supported the idea that the language for the parents to master was English and that they should speak it at home. The second language for the children was the language of mathematics. He held to the idea that mathematics was the gateway to success.

In keeping with Adolph's love of mathematics, the mathematics department at Essex County College offers the A. Shloming Mathematics Prize Exam that is open to all students at the college. This annual college level mathematics competition has been held for more than a decade. The participants are challenged with problems from algebra, geometry, trigonometry and elementary calculus. Being exposed to challenging and non-routine problems results in student enthusiasm for mathematics.

Adolph's son, Robert Shloming, sponsors this prize exam in honor of his father. Following his father's dedication to mathematics, Robert earned a Ph.D. from New York University and has taught in the Essex County College mathematics department for more than three decades. By offering this mathematics exam, Dr. Shloming hopes that the best and the brightest Essex County College students will share in the challenges and joys of this competition. But it is really more than a competition. One of the objectives is to create a community of students that will be enthusiastic and motivated to discuss mathematical problems and consider careers in mathematics or related fields.

I wish to thank the entire mathematics department for supporting this endeavor. Also I would like to extend my special gratitude to Prof. Ron Bannon. Without Ron's tireless efforts to develop, publicize, proofread and score the prize exam, Essex County College would not have this wonderful event. Last, but certainly not least, I want to congratulate all the participants who have taken a courageous step in experiencing a mathematics competition.

Good Luck to All, Dr. Robert Shloming