Exercise Seven: Math Miscellany

Definition. The *inverse tangent* or *arctangent* is denoted $\tan^{-1} x$ or $\arctan x$ and is the angle between $-\pi/2$ and $\pi/2$ whose tangent is equal to x.

Example 1. Since $\tan(\pi/4) = 1$, we also have $\arctan 1 = \pi/4$.

Example 2. However, $\tan(9\pi/4) = 1$, but $\arctan 1 = \pi/4$.

Question to Ponder 1. Should we really be anthropomorphizing an angle by using the word "whose" here?

Now we use the arctangent and some calculus to derive a wonderful series.

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\begin{aligned}
\frac{\pi}{4} &= \arctan 1 \\
&= \int_0^1 \frac{1}{1+x^2}\,dx && \text{Surely you remember that } \frac{d}{dx}\arctan x = \frac{1}{1+x^2}. \\
&= \int_0^1 \frac{1}{1-(-x^2)}\,dx && \text{Prepare to use the series } \frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots \\
&= \int_0^1 [1 - x^2 + x^4 - \dots]\,dx && \text{Substitute } u = -x^2 \text{ into the series above. Duh.} \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right]_0^1 && \text{Ever heard of the FTC?} \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots && \text{A moment of silence, please.}
\end{aligned}
$$

Question to Ponder 2. What on earth do the reciprocals of the odd natural numbers have to do with π?!?