Exercise Nine: Arrows and Functions

**Definition.** A function \( f : A \to B \) is bijective or a one-to-one correspondence if and only if \( f \) is injective and surjective.

**Notation.** If \( f \) is a bijection, we write \( f : A \xrightarrow{1-1} \text{onto} B \).

**Claim 1.** The function \( f : \mathbb{R} \to \mathbb{R}^+ \) given by \( f(x) = x^4 + 1 \) is not injective.

**Proof.** We must show that there exist \( a_1, a_2 \in \mathbb{R} \) such that \( f(a_1) = f(a_2) \) but \( a_1 \neq a_2 \).

Choose \( a_1 = 1 \) and \( a_2 = -1 \).

Then \( f(a_1) = 2 \) and \( f(a_2) = 2 \) but \( a_1 \neq a_2 \), so \( f \) is not injective. \( \square \)

**Claim 2.** The function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = 3x + 1 \) is surjective.

**Proof.** We must show that for every \( b \in \mathbb{R} \) [the codomain] there exists \( a \in \mathbb{R} \) [the domain] such that \( f(a) = b \).

Pick any \( b \in \mathbb{R} \).

Let \( a = (b - 1)/3 \).

Since \( b \in \mathbb{R} \) and because the reals are closed under subtraction and non-zero division, we know that \( (b - 1)/3 \in \mathbb{R} \), i.e., \( a \) is in the domain of \( f \).

Furthermore,

\[
\begin{align*}
f(a) &= f \left( \frac{b - 1}{3} \right) \\
&= 3 \cdot \frac{b - 1}{3} + 1 \\
&= b - 1 + 1 \\
&= b
\end{align*}
\]

as desired. \( \square \)