1 Word Problems

Word problems are the bane of mathematics. Last year I gave a colleague a word problem to check and she lamented that it was really tricky. She’s a very bright woman, and has an excellent command of the English language. In fact I seek her advice on many occasions, and she is well-respected for her mathematical abilities as well as her intimate fluency with written language. So, why the silly preamble? Basically to tell you that even we in math get confused by words, so please don’t despair, but please do try!

Words, however, are the foundation of mathematics. Without words, we would have never gotten to mathematical reason in the first place, nor we would have any need for numerical fluency. Words, after-all, is the main way we communicate with one another. Of course there are other ways in which we communicate, but it is words that make for the most lasting impressions.

Anaïs Nin\textsuperscript{2} once said, “Truth is something which can’t be told in a few words. Those who simplify the universe only reduce the expansion of its meaning.”

1.1 Vexing Non-Calculus Examples

Okay, here’s some pretty tough problems that may interest you.

1. An old car has to travel a two-mile route, uphill and downhill. Because it is old, the car can climb the first mile—the ascent—no faster than an average speed of fifteen miles per hour. How fast does the car have to travel the second mile—on the descent it can go faster, of course—in order to achieve an average speed of thirty miles per hour?\textsuperscript{3}

2. Given:

\[
\begin{align*}
|x| + x + y &= 10 \\
x + |y| - y &= 12
\end{align*}
\]

solve for \(x + y\).\textsuperscript{4}

\textsuperscript{1}This document was prepared by Ron Bannon (ron.bannon@gmail.com) using \LaTeX. \\
\textsuperscript{2}A famous French writer. \\
\textsuperscript{3}This problem was sent to Albert Einstein by his friend Wertheimer. Einstein wrote this in reply to Wertheimer, “Your letter gave us a lot of amusement. The first intelligence test fooled both of us (Bucky and me). Only on working it out did I notice that no time is available for the downhill run! …[deleted text] …Such drolleries show us how stupid we are!”

\textsuperscript{4}Students—and teachers—invariably struggle with this problem because it does not reflect the normal tone in which these two concepts (absolute value and systems of equations) are normally presented. It is here, however, where one’s mathematical mettle is tested. Some students solve the problem rabidly, while others take many frustrating hours before seeing the simplicity. But perhaps the entire point here is not necessarily solving the problem, but getting students interested in trying. One student in particular worked on this one problem for five hours, and then Eureka! …he solved it. The look on his face reflected a joy in realizing his hours of frustration were not for naught.
2 Guidelines For Solving Related-Rate Word Problems

1. Read the problem, they’re incredibly simple stories. Reading is perhaps the most fundamental activity in any academic environment and you need to understand what you read. You may also have to revisit word problems from prior courses.

2. Make a sketch of what’s happening in the problem. Label your sketch and be sure to write down any known relationships. For example, if they’re talking about the surface area of a cube, it would be a good idea to write down the algebraic relationship of a cube’s surface area. Variables (things that change) should be clearly understood at this stage.

3. Write down any derivatives given and any derivatives wanted. You’ll also need to find an algebraic/trigonometric relationship between the variables in those derivatives.

4. Differentiate this relationship with respect to time \( t \).

5. Now just plug in the known quantities and solve for the unknown.

Here’s a very simple example.

Find the rate of change of the distance between the origin and a moving point on the graph of \( y = x^2 + 1 \) if \( \frac{dx}{dt} = 2 \) centimeters per second and the point is at \((1.1, 2.21)\).

Details follow, but first think about the steps!

1. Okay, I read the problem. It looks simple enough.

2. Here’s a picture. The blue line is the distance between the origin and some point on the parabola—the point is moving constantly so the point on the parabola is not static. The origin does not change, but the length of the blue line changes as the point moves along the parabola. The point on the parabola is \((x, x^2 + 1)\) and the distance between this point and the origin is:

\[
d = \sqrt{(x - 0)^2 + (x^2 + 1 - 0)^2} \quad \Rightarrow \quad d^2 = x^2 + (x^2 + 1)^2, \quad x \in \mathbb{R}, \quad d \geq 1.
\]

![Figure 1](image.png)

Figure 1: This is how I see the problem.
3. Clearly they gave us \( \frac{dx}{dt} = 2 \), and they want \( \frac{dd}{dt} \) when the moving point is at (1.1, 2.21) on the parabola. Certainly, since \( x \) is getting bigger we would expect at this point that \( d \) is also getting bigger. The relationship between \( d \) and \( x \) from step 2 is:

\[
d^2 = x^2 + (x^2 + 1)^2, \quad x \in \mathbb{R}, \quad d \geq 1.
\]

4. Differentiate this relationship with respect to time \( t \).

\[
\frac{d}{dt}[d^2] = \frac{d}{dt}\left[x^2 + (x^2 + 1)^2\right]
\]

\[
2d \frac{dd}{dt} = [2x + 2(x^2 + 1)2x] \frac{dx}{dt}
\]

5. Now just plug in the known quantities and solve for the unknown. We have \( x = 1.1 \),

\[
d = \sqrt{(1.1)^2 + (1.1^2 + 1)^2} = \sqrt{6.0941}, \; \frac{dx}{dt} = 2.
\]

Plugging in we get:

\[
2\sqrt{6.0941} \frac{dd}{dt} = [2 \cdot 1.1 + 2 (1.1^2 + 1) 2 \cdot 1.1] 2
\]

\[
2\sqrt{6.0941} \frac{dd}{dt} = 23.848
\]

\[
2\sqrt{6.0941} \frac{dd}{dt} = \frac{23.848}{2\sqrt{6.0941}} \approx 4.483022
\]

So the answer is \[4.48\ \text{centimeters per second}\]

3 Examples

1. A particle moves along the curve \( y = \sqrt{1 + x^3} \). As it reaches the point (2, 3), the \( y \)-coordinate is increasing at a rate of 4 cm/sec. How fast is the \( x \)-coordinate of the point changing at that instant?

2. If a snowball melts so that its surface area decreases at a rate of 1 cm\(^2\)/min, find the rate at which the diameter decreases when the diameter is 10 cm?\(^5\)

3. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 3.44°/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is 60°?

\(^5\text{Hint: It’s okay to look up a geometry formula. They’re in your book!}\)
4 Solutions

1. A particle moves along the curve \( y = \sqrt{1 + x^3} \). As it reaches the point (2, 3), the \( y \)-coordinate is increasing at a rate of 4 cm/sec. How fast is the \( x \)-coordinate of the point changing at that instant?

**Work:** This will also be discussed in class.

\[
y = \sqrt{1 + x^3} \\
y = (1 + x^3)^{1/2} \\
\frac{dy}{dt} = \frac{3x^2}{2\sqrt{1 + x^3}} dx \\
\]

Now plugging in.

\[
\frac{dy}{dt} = \frac{3x^2}{2\sqrt{1 + x^3}} dx \\
4 = \frac{3 \cdot 2^2}{2\sqrt{1 + 2^3}} dx \\
4 = \frac{12}{2\sqrt{9}} \frac{dx}{dt} \\
4 = 2 \frac{dx}{dt} \\
2 = \frac{dx}{dt}
\]

The \( x \)-coordinate of the point is changing at a rate of \( 2 \text{ cm/sec} \) at that instant.

2. If a snowball melts so that its surface area decreases at a rate of 1 \( \text{cm}^2/\text{min} \), find the rate at which the diameter decreases when the diameter is 10 cm.\(^6\)

**Work:** This will also be discussed in class. The surface area of a sphere is:

\[ A = 4\pi r^2 \Rightarrow A = 4\pi \left( \frac{d}{2} \right)^2 = \pi d^2 \]

Differentiate with respect to \( t \).

\[
\frac{dA}{dt} = \pi d^2 \\
2\pi d \frac{dd}{dt}
\]

Now plugging in.

\[
2\pi d \frac{dd}{dt} = -1 = 2\pi 10 \frac{dd}{dt} \\
-1 = 20\pi \frac{dd}{dt} \\
\frac{1}{20\pi} = \frac{dd}{dt}
\]

\(^6\)Hint: It’s okay to look up a geometry formula. They’re in your book!
So the answer is

\[ -\frac{1}{20\pi} \text{ cm min} \]

3. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 3.44°/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is 60°?

**Work:** This will also be discussed in class. You should also convert all angles to radian.

\[
\frac{1}{2} \cdot 5 \cdot 4 \sin (\pi - \theta) \text{ m}^2 = A
\]
\[
10 \sin (\pi - \theta) \text{ m}^2 = A
\]
\[-10 \cos (\pi - \theta) \text{ m}^2 \frac{d\theta}{dt} = \frac{dA}{dt}
\]

Converting to radian.

\[
60^\circ = \frac{\pi}{3}, \quad \text{and} \quad \frac{d\theta}{dt} = \frac{3.44^\circ}{\sec} = \frac{\pi}{180^\circ} = \frac{3.44\pi}{180 \sec}
\]

Now plugging in.

\[
-10 \cos (\pi - \theta) \text{ m}^2 \frac{d\theta}{dt} = \frac{dA}{dt}
\]
\[
-10 \cos \left(\frac{\pi}{3}\right) \text{ m}^2 \cdot \frac{3.44\pi}{180 \sec} = \frac{dA}{dt}
\]
\[
-10 \left(-\frac{1}{2}\right) \text{ m}^2 \cdot \frac{3.44\pi}{180 \sec} = \frac{dA}{dt}
\]
\[
0.300196631343 \text{ m}^2 \sec \approx \frac{dA}{dt}
\]
Assignment:

1. Read §3.9.

2. You should be able to do the following problems.

   **3.9 Related Rates:** 2, 5, 8, 11, 13, 18, 24, 26, 27, 31, 36, 38, 42, 44.

   WebAssign problems, similar to the ones above, will be posted and you need to get started right away!

3. Again, don’t fall behind!