MTH 121 — Web Based Material
Essex County College — Division of Mathematics and Physics
Worksheet #14, Last Update July 16, 2010[1]

1 Hyperbolic Functions

Well, I can't say, "once again!" That is, the hyperbolic functions were not covered in MTH-119 or MTH-120, so we must begin afresh. So let's start with hyperbolic sine, abbreviated sinh, and hyperbolic cosine, abbreviated cosh.[2] They are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Their graphs follow:

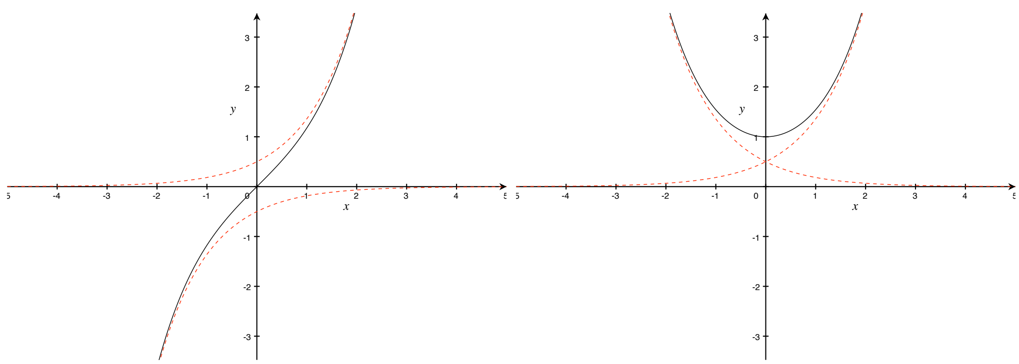

Figure 1: $y = \sinh x$ (left) and $y = \cosh x$.

The dashed red lines are not part of the graph, but I want to emphasize that these two functions are just simple combinations of e^x and e^{-x}. We'll discuss, in class, the dashed red lines on the above two graphs.

1.1 Some Questions?

1. What is the domain of the hyperbolic sine?

2. What is the range of the hyperbolic sine?

3. Is the hyperbolic sine *even*, *odd*, or *neither*?

4. What is the domain of the hyperbolic cosine?

[1]This document was prepared by Ron Bannon (`ron.bannon@gmail.com`) using LaTeX 2ε.
[2]The graph of the hyperbolic cosine is called a *catenary*, the shape of a hanging cable.

5. What is the range of the hyperbolic cosine?

6. Is the hyperbolic cosine *even*, *odd*, or *neither*?

7. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x.$$

8. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x.$$

9. Show that

$$\cosh^2 x - \sinh^2 x = 1.$$

10. If we define hyperbolic tangent as:

$$\tanh x = \frac{\sinh x}{\cosh x},$$

what is hyperbolic tangent in terms of e^x and e^{-x}?

By the way, Stewart's book has an interesting picture[3] that relates the point $P(\cosh t,\ \sinh t) = (x,\ y)$ on a hyperbola to the origin $O(0,\ 0)$ using the identity derived above, that is $\cosh^2 t - \sinh^2 t = 1$ or $x^2 - y^2 = 1$. Here's the graph of $x^2 - y^2 = 1$, which is a hyperbola.[4]

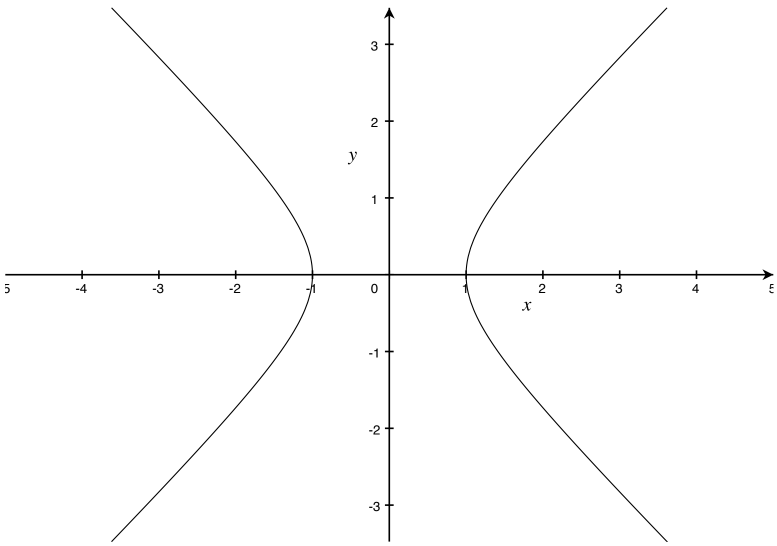

Figure 2: Graph of $x^2 - y^2 = 1$, which is a hyperbola.

Hence, the reason why they're called hyperbolic, just as the trigonometric functions are often called circular. That is, the point $P(\cos t,\ \sin t) = (x,\ y)$ on a circle is related to the origin $O(0,\ 0)$ using the identity $\cos^2 t + \sin^2 t = 1$ or $x^2 + y^2 = 1$.

[Example:] Pick the point $\left(\sqrt{2},\ 1\right)$ on the hyperbola, $x^2 - y^2 = 1$, and determine the exact value of t.[5]

Work: Let's find t first.

$$
\begin{array}{rcll}
\sinh t & = & 1 & \\
\dfrac{e^t - e^{-t}}{2} & = & 1 & \\
e^t - e^{-t} & = & 2 & \text{multiply both sides by } e^t \\
e^{2t} - 1 & = & 2e^t & \quad \text{solve for zero} \\
e^{2t} - 2e^t - 1 & = & 0 & \text{let } u = e^t \text{ and note that } u > 0 \\
u^2 - 2u - 1 & = & 0 & \text{solve for } u \text{ using the quadratic formula} \\
u & = & 1 + \sqrt{2} & \quad \text{now solve for } t, \text{ where } t = \ln u. \\
t & = & \ln\left(1 + \sqrt{2}\right) &
\end{array}
$$

[3] The parametric variable, t, when dealing with the trigonometric functions, represents the angle. Interestingly enough, the t is also related to the area of the sector of the unit circle; likewise, the same is true for the hyperbolic functions, but this time the area is not a sector, but the bounded region.

[4] The circular functions relate the point $P(\cos t,\ \sin t) = (x,\ y)$ on a unit circle, $x^2 + y^2 = 1$ to the origin $O(0,\ 0)$.

[5] Again, the point $P(\cosh t,\ \sinh t) = (x,\ y)$ is on $x^2 - y^2 = 1$.

Now, using this value of t, verify $\cosh t = \sqrt{2}$.

$$\begin{aligned}
\cosh t &= \sqrt{2} \\
\frac{e^t + e^{-t}}{2} &= \sqrt{2} \\
\frac{e^{\ln\left(1+\sqrt{2}\right)} + e^{-\ln\left(1+\sqrt{2}\right)}}{2} &= \sqrt{2} \\
\frac{\left(1+\sqrt{2}\right) + \left(1+\sqrt{2}\right)^{-1}}{2} &= \sqrt{2} \\
\frac{\left(1+\sqrt{2}\right) + \left(\sqrt{2}-1\right)}{2} &= \sqrt{2} \\
\frac{2\sqrt{2}}{2} &= \sqrt{2} \\
\sqrt{2} &= \sqrt{2}
\end{aligned}$$

So it should be clear why the term hyperbolic is used, and the reason that we see *sine cosine* and *tangent* in these hyperbolic functions is mainly due to the fact that the identities they generate is reminiscent of the trigonometric identities.

1.2 Inverses

From what we know about inverses, it is clear that the hyperbolic sine is invertible, but the hyperbolic cosine is not. However, just like the trigonometric functions, we are going to restrict the domain of the hyperbolic cosine to make it a one-to-one function. For sake of argument, let's restrict the domain of the hyperbolic cosine to $x \geq 0$.

As an example, let's try finding $\sinh^{-1} x$.

$$\begin{aligned}
\sinh x &= y \\
\sinh y &= x \\
\frac{e^y - e^{-y}}{2} &= x \\
e^y - e^{-y} &= 2x \qquad \text{multiply both sides by } e^y \\
e^{2y} - 1 &= 2xe^y \qquad \text{solve for zero} \\
e^{2y} - 2xe^y - 1 &= 0 \qquad \text{let } u = e^y \\
u^2 - 2xu - 1 &= 0 \\
u &= \frac{2x \pm \sqrt{4x^2+4}}{2} \qquad \text{quadratic formula} \\
e^y &= x + \sqrt{x^2+1} \qquad \text{note the } \pm \text{ is now } + \\
y &= \ln\left(x + \sqrt{x^2+1}\right)
\end{aligned}$$

Side Note: Why only the $+$ is used from the expression $x \pm \sqrt{x^2+1}$, because we know that $e^y > 0$, hence $x \pm \sqrt{x^2+1} > 0$, and since $x < \sqrt{x^2+1}$ for all x we must exclude $x - \sqrt{x^2+1} < 0$.

1.3 What You Need to Know?

Certainly you should be able to recall the definitions for the hyperbolic sine and cosine, and then follow what was presented in this sheet so far. When doing the homework it is okay to use the following results (no need to memorize them).

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$
2. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$
3. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tanh\ x) = \mathrm{sech}^2\ x$
4. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{csch}\ x) = -\mathrm{csch}\ x\ \coth\ x$
5. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\ x) = -\mathrm{sech}\ x\ \tanh\ x$
6. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\coth\ x) = -\mathrm{csch}^2\ x$
7. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$
8. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1,\ \infty)$
9. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad x \in (-1,\ 1)$
10. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$
11. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}$
12. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$
13. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{csch}^{-1} x\right) = -\dfrac{1}{|x|\sqrt{1+x^2}}$
14. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{sech}^{-1} x\right) = -\dfrac{1}{x\sqrt{1-x^2}}$
15. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2}$

Believe it or not, the results above can be obtained by using the definitions of the hyperbolic sine and cosine alone. However, you also need to be familiar with some basic identities from trigonometry that relate sine and cosine to the other four trigonometric functions. The hyperbolic functions have the same fundamental identities.

1.4 Problems

1. Prove the identity.

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

2. Prove the identity.

$$\sinh 2x = 2 \sinh x \cosh x$$

3. If $\sinh x = 3/4$, find the values of $\cosh x$ and $\tanh x$.

4. Find the derivative.

$$y = x \cosh x$$

5. Find the derivative.

$$y = \sinh x \cosh x$$

6. Find the derivative.

$$y = x^2 \sinh^{-1}(2x)$$

7. Evaluate.

$$\lim_{x \to \infty} \frac{\sinh x}{e^x}$$

1.5 Solutions

Will be done in class.

Assignment:

1. Read §3.11.
2. No problems will be assigned from this section, but you should nonetheless be familiar the hyperbolic functions and their derivatives.