MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #1 — Sakai Web Project Material

1 Preliminaries

- All students at ECC should have an email address on file with the school. I use this email address to generate a group email list and you should have received an email from me informing you about our group website. If you did not receive an email from me you can still visit our group website at `http://groups.google.com/group/mth-122-fall-2009` and once there you can join the group. It is important that all registered students join this group and download/print posted files.

2 General Differential Equations

When you finished the first course in calculus you basically covered differentiation and integration. You should have an intuitive idea what it means to differentiate and to integrate, and I strongly suggest you review these topics if you're confused or just a bit rusty. Now, however, we will review these *old* topics in a *new* and possibly confusing way. Yes, the scary sounding newness is actually an introduction to *differential equations*, but this topic will basically be used to review the process of differentiation and integration.

So what is a differential equation anyway? Well, a differential equation is basically an equation that contains one or more derivatives. For example,

$$y' + y \tan x = \cos^2 x$$

is a differential equation where y' represents a derivative of y with respect to x. And y, although confusing at first, represents a function of x. Some may even write y explicitly as a function of x by writing this differential equation as,

$$y'(x) + y(x) \tan x = \cos^2 x.$$

Yes, that is scary, but suppose I were to tell you that

$$y = \sin x \cos x - \cos x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

is a solution to the above differential equation, just as I might tell a beginning algebra student that $x = 2$ is a solution to the *linear* equation, $2x - 1 = 3$. Certainly you'd have no problem with verifying the linear case, but could you also verify that this given y is actually a solution to our differential equation?

[1]This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using $\LaTeX\,2_\varepsilon$. Last revised September 8, 2009.

Work: Verify that $y = \sin x \cos x - \cos x$ is a solution to the differential equation $y' + y \tan x = \cos^2 x$.

The graph below may look unfamiliar, but it is simply a graph of $y = \sin x \cos x - \cos x$, and a collection of small line segments drawn at various points $(x,\ y)$, where the slopes of these line segments is the value of y'. For example, at the point $(0,\ 0)$ we get

$$y' + 0 \tan 0 = \cos^2 0 \quad \Rightarrow \quad y' = 1.$$

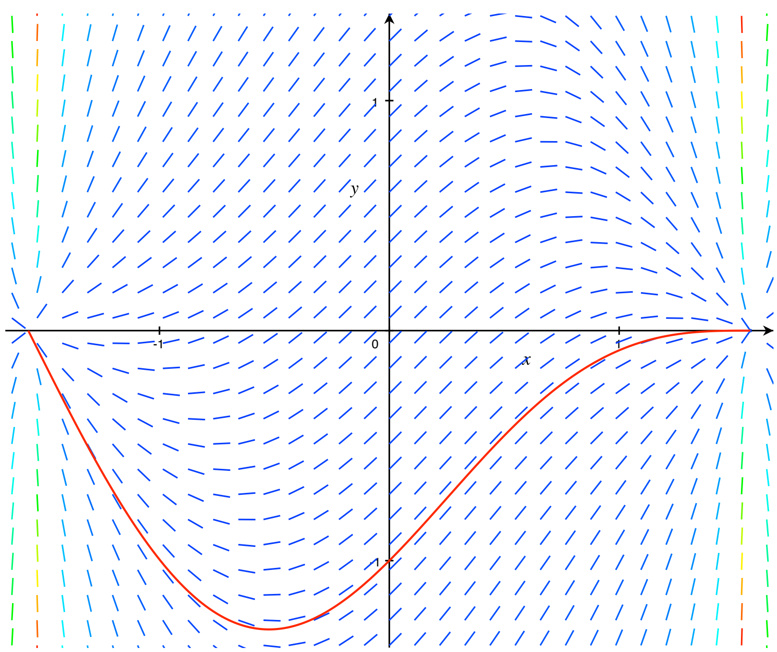

Figure 1: The direction field of $y'(x) + y(x) \tan x = \cos^2 x$, and one solution.

Doing this by hand can be quite difficult, and almost certainly frustrating. Although the book will ask you to do this by hand, I strongly suggest you learn to do this on a computer. If you do a simple search on the web for *direction field* you'll probably find many free programs that can easily do this.

In general, we will be given a first-order[2] differential equations of the form

$$y' = F(x,\ y).$$

For example, suppose we're given

$$y' = -\frac{x}{y}$$

and then asked to find its solution? Although this is a *very* simple example, I believe that most students wouldn't even know where to begin. However, if I gave you the solution(s), I think you could at least verify it. I will say though, that this is a familiar example, and if your memory is good you may even recall that this was done last semester. For now, let's just examine the *direction field*[3]. Again, the *direction field* is just a bunch of short line segments drawn at various points, that have slope:

$$y' = F(x,\ y),$$

or in this case,

$$y' = -\frac{x}{y}.$$

Here you should try to make sense out of the line segments. For example, what happens when

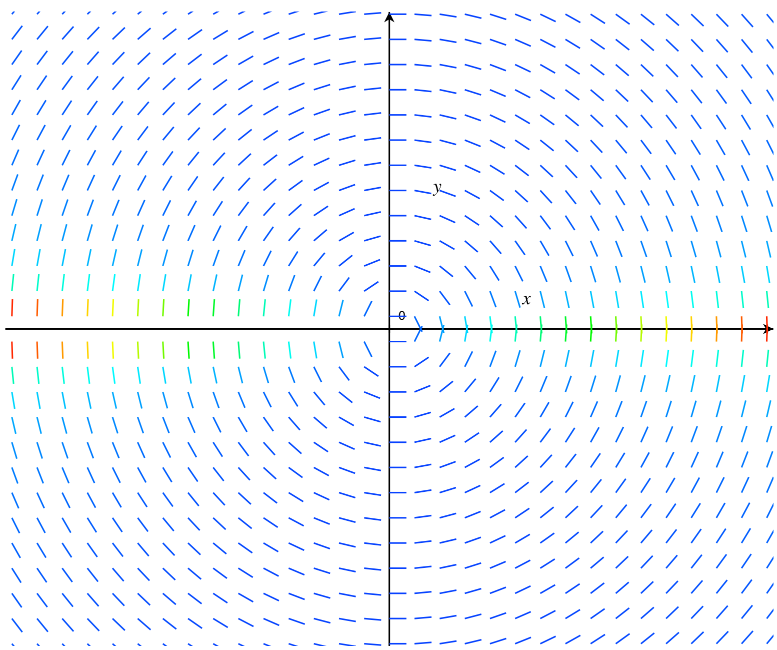

Figure 2: Direction field for $y' = -x/y$.

$x = 0$, or when $y = 0$. Although I'd hate to see someone draw each line segment by hand, you should be able to identify (verbally) why the segments have a particular slope. That is, why it's negative/positive, zero, small, large, or some particular value. In our example you should be able to circle every instance where $y = x$ and where $y = -x$. Now, let's try to draw a solution, by starting somewhere and moving slowly in the direction of the field. Software can do this to! Looking at these solutions, can you make a guess as to what the *general* solution is? Yes,

[2]Equations with first derivatives only.

[3]Also called a *slope field*.

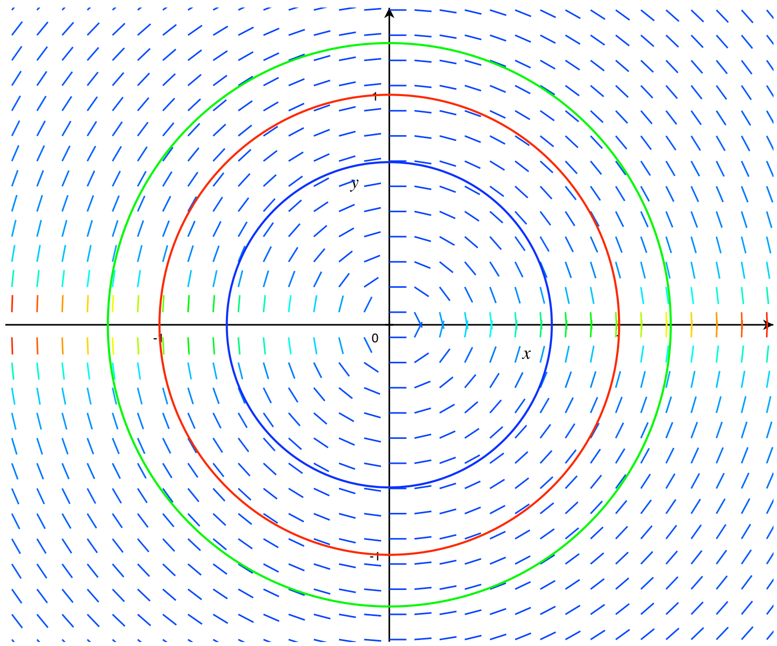

Figure 3: Three, among an infinite number of solutions.

they're *circles*, of the form:

$$x^2 + y^2 = r^2.$$

Work: Verify that $y^2 + x^2 = r^2$ is a solution to the differential equation $y' = -x/y$.

When we're given a differential equation

$$y' = F(x,\ y),$$

we may also be given an *initial* condition. For example in the above case, we may be told that

$$y' = -x/y, \quad y(-1) = 0.$$

Here the $y(-1) = 0$ is the *initial* condition.

Work: If this *initial* condition is to be met, that is $y(-1) = 0$, what is the particular solution to our differential equation $y' = -x/y$?

Example: The differential equation[4]

$$y' = \frac{y^2 - 1}{2},$$

has a solution of the form

$$y = \frac{1 + ke^x}{1 - ke^x}.$$

Verify, that for every value of k, that this is actually a solution. And then given the initial condition that $y(0) = 2$, find the particular solution that satisfies this initial condition.

Work:

Finally, here's the graph of the above problem. Can you look at this graph and find an obvious solution that can be easily verified?

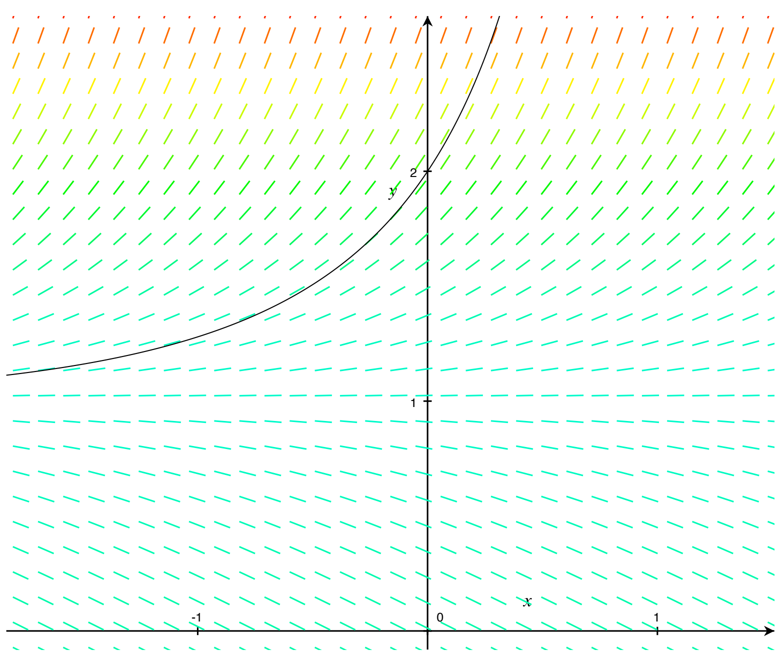

Figure 4: Direction field for $y' = \left(y^2 - 1\right)/2$ and the solution that contains $(0,\ 2)$.

Now it's time to reinforce what was just introduced by reading sections 9.1 and 9.2 of your textbook, then start the WebAssigns assignments for these sections. **Don't fall behind.**

[4]A differential equation of the form $y' = f(y)$ in which the independent variable is missing, is called *autonomous*.