MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #2 — Sakai Web Project Material

1 Euler's Method

Direction fields, as I have been drawing them, are best done on a computer. They're really tedious when done by hand, but once drawn they can give you a fairly good idea of what solutions may look like. In the prior examples I actually knew the solutions, so I was able to graph those superimposed over the direction field. Now, let's say I don't know the solution, but I still want to approximate the graph of the solution over the direction field. I'm using a program called Grapher,[2] and it allows me to draw both the direction field for a given differential equation, and several types of approximate solutions. One simple technique that the book mentions is Euler's method, and that's what I will concentrate on here. Let's look at an example,

$$y' = x + y, \quad y(0) = 1,$$

where I will have Grapher plot the *direction field*, and a numerical solution, through the point $(0,\ 1)$, using Euler's method with a step size of 0.1.

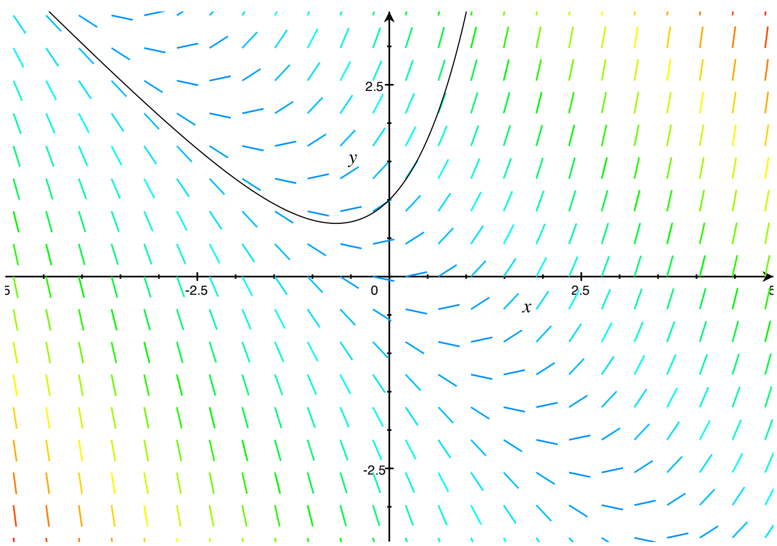

Figure 1: Direction field for $y' = x + y$, and a numerical solution that contains $(0,\ 1)$.

Fact is, Euler's method is just a visual, and I doubt seriously that anyone would be able to tell me the exact solution even after looking at the above image. However, I did the nasty work of finding the solution, and it's

$$y = 2e^x - x - 1.$$

You should be able to verify that this is a solution and satisfies the initial condition. But I really want to emphasize that Euler's method is an approximate visualization, so let's now look at how

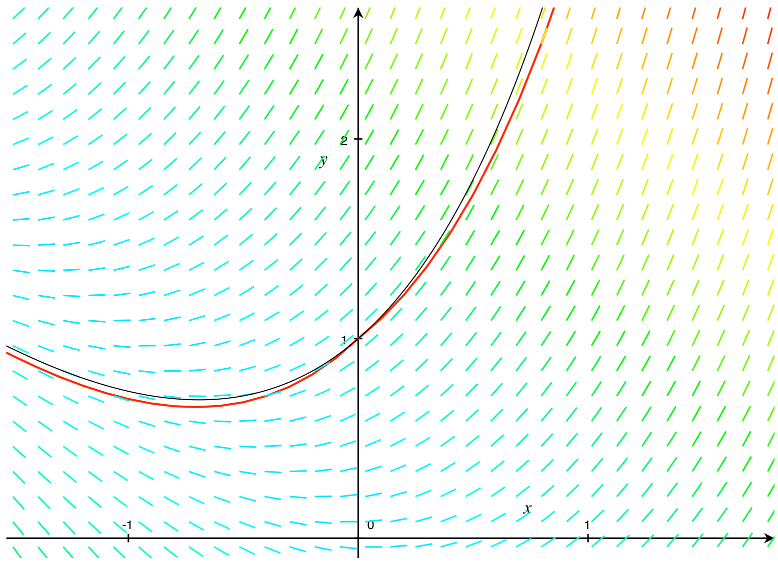

Figure 2: Direction field of $y' = x + y$, Euler's fit, and actual solution.

well Euler's method fits. The basic idea is to start at the initial condition, compute the slope, and then use a simple line to move to the next point. Since we're using a step size of $h = 0.1$, we need to move from 0, to 0.1, to 0.2 until we get to some desired value. For example, let's start at 0 and move to 1—that's going to take 10 iterations to get there. Using simple lines, we get:

$$\begin{aligned}
y_1 &= y_0 + h \cdot y'(x_0, y_0) && (1)\\
y_2 &= y_1 + h \cdot y'(x_1, y_1) && (2)\\
y_3 &= y_2 + h \cdot y'(x_2, y_2) && (3)\\
y_4 &= y_3 + h \cdot y'(x_3, y_3) && (4)\\
y_5 &= y_4 + h \cdot y'(x_4, y_4) && (5)\\
y_6 &= y_5 + h \cdot y'(x_5, y_5) && (6)\\
y_7 &= y_6 + h \cdot y'(x_6, y_6) && (7)\\
y_8 &= y_7 + h \cdot y'(x_7, y_7) && (8)\\
y_9 &= y_8 + h \cdot y'(x_8, y_8) && (9)\\
y_{10} &= y_9 + h \cdot y'(x_9, y_9) && (10)
\end{aligned}$$

You should note from the graph that the exact solution $y = 2e^x - x - 1$ and Euler's method seem to move apart as we move away from the initial starting position $(0,\ 1)$. Using Euler's method we get $(1.0,\ 3.187485)$, and the exact solution we get approximately $(1.0,\ 3.436564)$. which differs by about 0.25. As you may imagine, taking smaller step sizes will improve our approximations, but can be horribly tedious to compute. Again, doing this on a computer is encouraged, where changing the step size is trivial. However, computers can make awful mistakes as well and you'll almost always have to use your head.

[1]This document was prepared by Ron Bannon (ron.bannon@mathography.org) using LaTeX 2ε. Last revised September 8, 2009.

[2]Mac OS X only, but similar programs are available for UNIX, LINUX, and Windows.

I'm not suggesting that you compute these points by hand, but you might be interested in doing this just to see how tedious it can become.[3]

2 Separable Variables

Solving some first-order differential equations can be easily done, especially if we have an equation of this form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(x\right) f\left(y\right),$$

where we rewrite it as,

$$\frac{1}{f\left(y\right)}\ \mathrm{d}y = g\left(x\right)\ \mathrm{d}x,$$

and then integrate both sides,

$$\int \frac{1}{f\left(y\right)}\ \mathrm{d}y = \int g\left(x\right)\ \mathrm{d}x.$$

Much of what follows will be a review of simple integration. So let's start with a simple problem, where we're given,

$$y' = \frac{x}{y}, \quad y\left(0\right) = -3.$$

Let's not worry about the initial condition just now, but we do need to see the separation of the variables, as follows.

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y},$$

where we rewrite it as,

$$y\ \mathrm{d}y = x\ \mathrm{d}x,$$

and then integrate both sides,

$$\int y\ \mathrm{d}y = \int x\ \mathrm{d}x.$$

Of course we'll get a constant of integration on each side, but we'll put all constants on the right. As I hope you can see, this is a very simple integration. Here's what we get:

$$\begin{aligned}
\int y\ \mathrm{d}y &= \int x\ \mathrm{d}x \\
\frac{y^2}{2} + k_1 &= \frac{x^2}{2} + k_2 \\
\frac{y^2}{2} &= \frac{x^2}{2} + k_3 \\
\frac{y^2}{2} - \frac{x^2}{2} &= k_3 \\
y^2 - x^2 &= k_4
\end{aligned}$$

Now to find k_4 we use the fact that $y\left(0\right) = -3$, to get

$$y^2 - x^2 = k_4 \quad \Rightarrow \quad 9 = k_4.$$

Viola! Our solution is

$$y\left(x\right) = -\sqrt{9 + x^2}.$$

[3] Here they are: (0.100, 1.100), (0.200, 1.220), (0.300, 1.362), (0.400, 1.528), (0.500, 1.721), (0.600, 1.943), (0.700, 2.197), (0.800, 2.487), (0.900, 2.816), (1.000, 3.187).

Here's the *direction field*, our particular solution [RED], and Euler's method with $h = 0.5$ [BLUE]. Using technology is important, and having the ability to graph direction fields along with plotting numerical solutions can offer much insight.

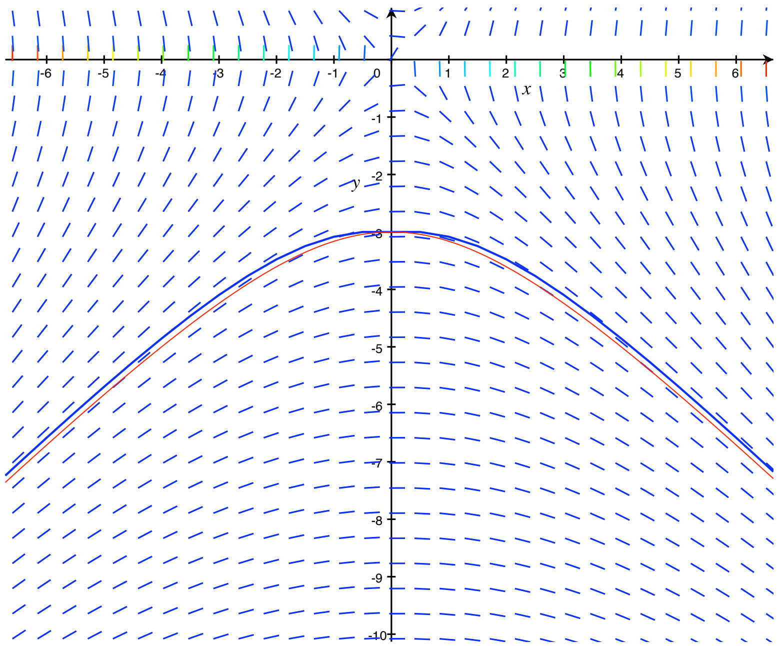

Figure 3: Direction field of $y' = x/y$, Euler's fit, and actual solution.

Let's continue with some examples.

1. Find the solution of the differential equation that satisfies the given initial condition.

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos x}{1+y^2}, \quad y\,(0) = 1.$$

   Work:

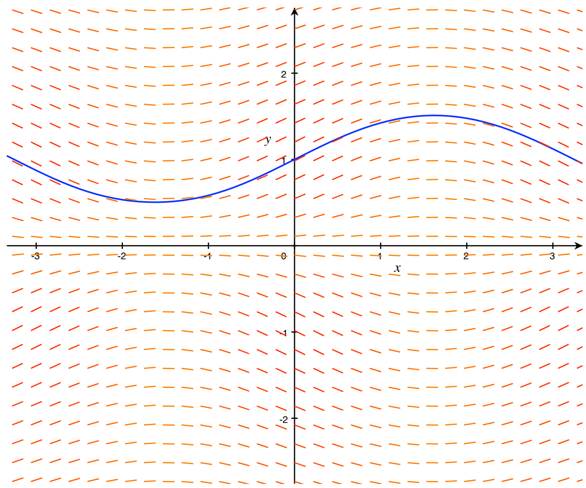

Figure 4: Let's label this graph together.

2. Find an equation of the curve that passes through the point $(0,\ 1)$ and whose slope at $(x,\ y)$ is xy.

   Work:

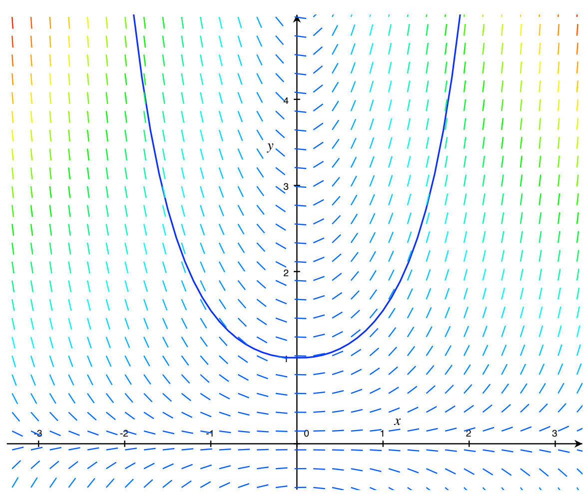

Figure 5: Let's label this graph together.

3. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$. Find the particular solution for $y(0) = 0$.

Work:

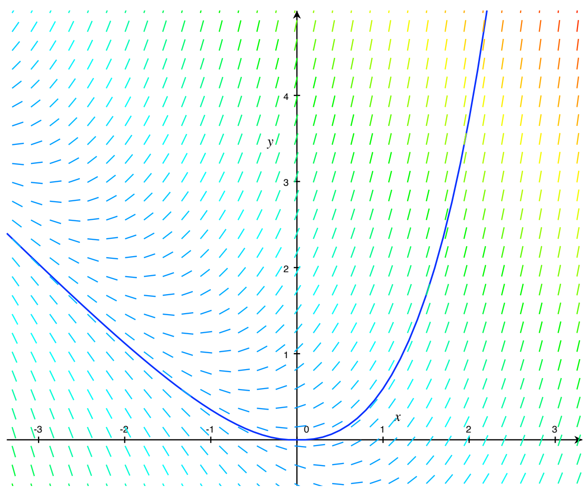

Figure 6: Let's label this graph together.

4. Find the function $f(x)$ such that $f'(x) = f(x)(1 - f(x))$ and $f(0) = 1/2$.[4]

Work:

[4]This problem's a bit too tricky at this point in time, and I'll give you a really *BIG hint* in class. This general method will be cover more throughly later in the semester.

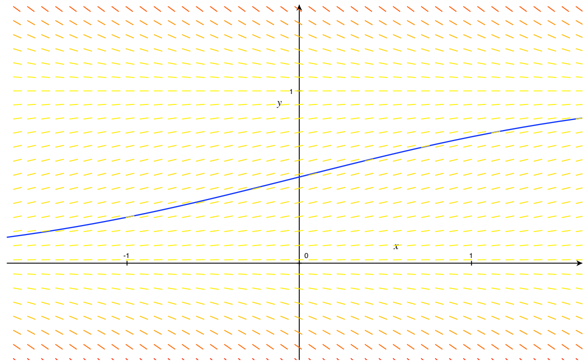

Figure 7: Let's label this graph together.

Here's the answers.

1. $\ln y + \frac{y^2}{2} = \sin x + \frac{1}{2}$
2. $y = e^{\frac{x^2}{2}}$
3. $y = e^x - 1 - x$
4. $y = \frac{e^x}{1+e^x}$

3 Modeling

A really simple population growth model would be

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky,$$

where t represents time and y represents the population as a function of t. The $k > 0$ is some growth constant. However, this is not a realistic model because it does account for factors[5] that limit growth. What's nice about this model is that it is very easy to solve for $y(t)$. Let's proceed.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= ky \\
\frac{1}{y}\,\mathrm{d}y &= k\ \mathrm{d}t \\
\int \frac{1}{y}\,\mathrm{d}y &= \int k\ \mathrm{d}t \\
\ln y &= kt + C \qquad y \text{ is always positive here.} \\
y &= e^{kt+C}
\end{aligned}$$

So if this is not realistic, what is? Well, a Belgian mathematician by the name Pierre-François Verhuist (1804-1849) proposed the following model,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{A}\right),$$

[5] Competition for limited resources, competition from other animals, and even environmental factors.

where t represents time and y represents the population as a function of t. The $k > 0$ is some growth constant, and the $A > 0$ is the carrying capacity. This equation is referred to as a *logistic differential equation*. We can use separation of variables, but the integration will prove too difficult at this time. I'll do the dirty integration work for now, but you should verify it!

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{A}\right) \quad \Rightarrow \quad y(t) = \frac{A}{1 - e^{-kt}/C}$$

As an example, suppose we have 10,000-acre forest, and it's able to support 1,000 deer (that's the carrying capacity). Also assume that the population follows the logistic model, where k is 0.4 per year. Find the deer population if the initial (beginning) population of deer is 100. How long will it take the population to reach 500?[6]

Work:

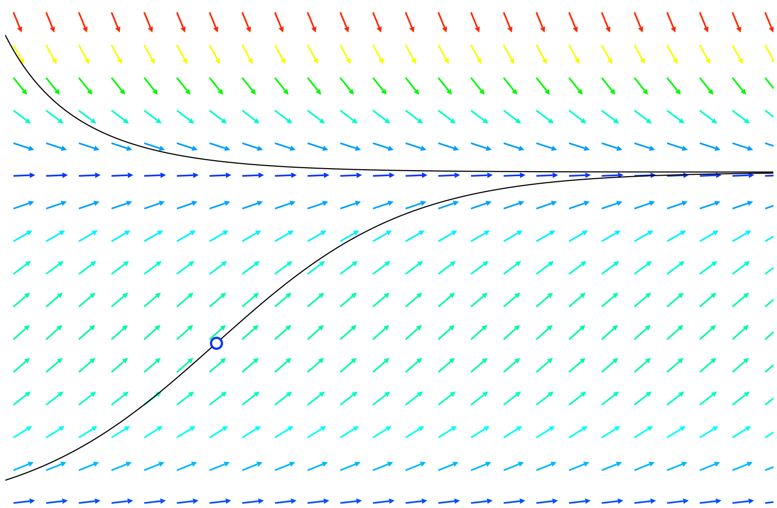

Figure 8: Let's label this graph together.

Now it's time to reinforce what was just introduced by reading sections 9.3 and 9.4 of your textbook, then start the WebAssigns assignments for these sections. **Don't fall behind.**

[6] $C = -1/9$, and the initial population of 100 deer will grow to 500 dear after $t \approx 5.5$ years.