MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #3 — Sakai Web Project Material

1 Integration by Parts

Let's start with an example where we're asked to verify:

$$\int xe^x \, \mathrm{d}x = xe^x - e^x + C.$$

As you know, we basically just have to check that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x - e^x + C\right) = xe^x.$$

Here goes:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x - e^x + C\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(C\right) \\
&= \left(xe^x + e^x\right) - \left(e^x\right) + (0) \\
&= xe^x
\end{aligned}$$

The main problem here is not the checking, but it is the actual process of finding the antiderivative. So far we've only used one real method, that was u-substitution, and although it is an important method it would fail miserably in this particular case. Another technique known as *integration by parts* uses the *product rule*. As you recall, the product rule is:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u'v + uv',$$

where u and v are functions of x and u' and v' are the derivatives with respect to x. We can rewrite this product rule as follows:

$$uv' = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - u'v,$$

and then integrate both sides,

$$\int uv' \, \mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv) \, \mathrm{d}x - \int u'v \, \mathrm{d}x.$$

Some may argue, rightfully so, that this is making our integrations more difficult, because now, instead of one integral (left side) we now have two (right side). However it should be noted that:

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(uv) \, \mathrm{d}x = uv + C.$$

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Furthermore, we can hold off on using the constant of integration until the end. That is, we should only write down C when all integrations are done on the right side. So the method finally looks like this:

$$\int uv' \, \mathrm{d}x = uv - \int u'v \, \mathrm{d}x.$$

If we have a definite integral, the above becomes:

$$\int_a^b uv' \, \mathrm{d}x = uv]_a^b - \int_a^b u'v \, \mathrm{d}x.$$

A simplification in both cases above is to recognize that $v' \, \mathrm{d}x = \mathrm{d}v$ and $u' \, \mathrm{d}x = \mathrm{d}u$. Finally we have for indefinite integrals:

$$\boxed{\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u}.$$

If we have a definite integral, the above becomes:

$$\boxed{\int_a^b u \, \mathrm{d}v = uv]_a^b - \int_a^b v \, \mathrm{d}u}.$$

Let's return to our first example. This time we are going to use the method described above to find the antiderivative. Basically we need to find a u and a $\mathrm{d}v$. As experience has shown me, the inexperienced will have many choices to consider, but as you become more experienced your choices will be reduced. And I generally believe too much choice is a recipe for disaster. Let's proceed with the following suggestions:

$$u = x \qquad \text{and} \qquad \mathrm{d}v = e^x \, \mathrm{d}x,$$

where this easily leads to (forgetting about C),

$$\mathrm{d}u = \mathrm{d}x \qquad \text{and} \qquad v = e^x.$$

Now, using these suggestions, lets try to rewrite our original example.

$$\begin{aligned} \int xe^x \, \mathrm{d}x &= xe^x - \int e^x \, \mathrm{d}x \\ &= xe^x - e^x + C \end{aligned}$$

It's really not so bad, as long as you know where to start. And initially I'll tell you where to start, and with experience you'll need to start yourself.

Example 1 Use $u = x$ and $\mathrm{d}v = \cos x \, \mathrm{d}x$.

$$\int x \cos x \, \mathrm{d}x$$

Example 2 Use $u = \ln x$ and $\mathrm{d}v = \mathrm{d}x$.

$$\int \ln x \, \mathrm{d}x$$

Example 3 Use $u = \ln x$ and $\mathrm{d}v = x^6\ \mathrm{d}x$.

$$\int x^6 \ln x\ \mathrm{d}x$$

Example 4 Use $u = x^2$ and $\mathrm{d}v = \sin 4x\ \mathrm{d}x$.

$$\int x^2 \sin 4x\ \mathrm{d}x$$

Example 5 Use $u = \cos x$ and $\mathrm{d}v = \cos x\ \mathrm{d}x$. You can also rewrite $\cos^2 x$ in terms of $\cos 2x$, which is *much* easier.

$$\int \cos^2 x\ \mathrm{d}x$$

Okay the last one is a bit tough, but you should at least be able to get here:

$$\int \cos^2 x\ \mathrm{d}x = \cos x \sin x + \int \sin^2 x\ \mathrm{d}x.$$

But then what? For a hint, let $\sin^2 x = 1 - \cos^2 x$. Taking the hints.

$$\begin{aligned}
\int \cos^2 x\ \mathrm{d}x &= \cos x \sin x + \int \sin^2 x\ \mathrm{d}x \\
&= \cos x \sin x + \int 1 - \cos^2 x\ \mathrm{d}x \\
&= \cos x \sin x + x - \int \cos^2 x\ \mathrm{d}x \\
2\int \cos^2 x\ \mathrm{d}x &= \cos x \sin x + x + C \\
\int \cos^2 x\ \mathrm{d}x &= \frac{\cos x \sin x + x}{2} + C
\end{aligned}$$

Now, as suggested in the beginning, try using a double angle formula instead.

$$\begin{aligned}
\int \cos^2 x\ \mathrm{d}x &= \int \frac{\cos 2x + 1}{2}\ \mathrm{d}x \\
&= \frac{\sin 2x}{4} + \frac{x}{2} + C \\
&= \frac{2\sin x \cos x}{4} + \frac{x}{2} + C \\
&= \frac{\cos x \sin x + x}{2} + C
\end{aligned}$$

2 Additional Examples

1. Use integration by parts to verify.

$$\int x e^{-x}\ \mathrm{d}x = -xe^{-x} - e^{-x} + C$$

2. Use integration by parts to verify.

$$\int \arcsin x \, \mathrm{d}x = x \arcsin x + \sqrt{1 - x^2} + C$$

3. Use u-substitution to verify.

$$\int_0^1 \frac{x^3}{\sqrt{4 + x^2}} \, \mathrm{d}x = \frac{16 - 7\sqrt{5}}{3}$$

   Now try integration by parts by letting $u = x^2$ and $\mathrm{d}v = \dfrac{x}{\sqrt{4 + x^2}} \, \mathrm{d}x$

4. Use integration by parts to verify.

$$\int_1^2 x^3 \ln x \, \mathrm{d}x = \ln 16 - \frac{15}{16}$$