1 Integration of Rational Functions Using Partial Fractions

In the past it was far more usual to simplify or combine fractions than it was to expand them. However, it is sometimes easier to break an integrand up into simpler problems, and for rational expressions we have several forms to recognize. And I want to again *emphasize* that recognition does not necessarily mean *memorization*.

Here's a partial list that should be recognized, and I am putting a $\boxed{\text{boxes}}$ around those that may be helpful when integrating rational functions:

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1}x\right) = \dfrac{1}{\sqrt{1+x^2}}$. Which of course leads to:

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \sinh^{-1}x + C$$

2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}x\right) = \dfrac{1}{\sqrt{x^2-1}}$. Which of course leads to:

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \cosh^{-1}x + C$$

3. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}x\right) = \dfrac{1}{1-x^2}$. Which of course leads to:

$$\int \frac{1}{1-x^2}\,\mathrm{d}x = \tanh^{-1}x + C$$

4. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{csch}^{-1}x\right) = -\dfrac{1}{|x|\sqrt{1+x^2}}$. Which of course leads to:

$$\int \frac{1}{|x|\sqrt{1+x^2}}\,\mathrm{d}x = -\operatorname{csch}^{-1}x + C$$

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{sech}^{-1}x\right) = -\dfrac{1}{x\sqrt{1-x^2}}$. Which of course leads to:

$$\int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\operatorname{sech}^{-1}x + C$$

6. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1}x\right) = \dfrac{1}{1-x^2}$. Which of course leads to:

$$\int \frac{1}{1-x^2}\,\mathrm{d}x = \coth^{-1}x + C$$

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[1]This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised January 10, 2009.

I think, however, that these should probably be memorized, at least for an exam.

1.
$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

2.
$$\boxed{\int \frac{1}{1+x^2} \, dx = \arctan x + C}$$

3.
$$\boxed{\int \frac{1}{x} \, dx = \ln|x| + C}$$

2 The Pre-Calulus of Partial Fraction Decomposition

You'll have a rational function in lowest terms of the form

$$\frac{P(x)}{Q(x)}.$$

Then you'll need to follow the steps below.

1. If the degree of the numerator is greater than or equal to the denominator, long divide. You will be left with a rational function. For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{3x^2 - 2x - 1}.$$

2. If the degree of the numerator is less than the degree of the denominator, then factor the denominator into linear factors or irreducible[2] quadratic factors. An irreducible quadratic factor is a quadratic that cannot not be factored into linear factors with real coefficients. Example continued from above.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{(3x + 1)(x - 1)}.$$

3. Assign each linear factor $(px + q)^n$ the sum of n partial fractions:

$$\frac{A_1}{(px + q)^1} + \frac{A_2}{(px + q)^2} + \frac{A_3}{(px + q)^3} + \cdots + \frac{A_n}{(px + q)^n}$$

Continuing with the example above.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{(3x + 1)(x - 1)} = 2x + 3 + \frac{k_1}{(3x + 1)} + \frac{k_2}{(x - 1)}$$

Of course we will need to determine the values of the constants k_1 and k_2.

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[2]A quadratic polynomial $ax^2 + bx + c$ is called irreducible if it cannot be factored into linear factors without using complex numbers. *e.g.* $x^2 + 1 = (x - i)(x + i)$

4. Assign each irreducible quadratic factor $\left(ax^2 + bx + c\right)^m$ the sum of m partial fractions:

$$\frac{B_1 x + C_1}{\left(ax^2 + bx + c\right)^1} + \frac{B_2 x + C_2}{\left(ax^2 + bx + c\right)^2} + \frac{B_3 x + C_3}{\left(ax^2 + bx + c\right)^3} + \cdots + \frac{B_m x + C_m}{\left(ax^2 + bx + c\right)^m}.$$

Again you'll need to determine these constants

5. This looks nightmarish, I know, but the algebra is really quite easy. Of course, let's look at some examples.

2.1 Preliminary Examples

1. Decompose into partial fractions.
$$\frac{4x - 13}{2x^2 + x - 6}$$

Work: The first step is to break the fraction into two parts.

$$\frac{4x - 13}{2x^2 + x - 6} = \frac{A}{x + 2} + \frac{B}{2x - 3}$$

Now add the fractions on the right hand side together.

$$\frac{4x - 13}{2x^2 + x - 6} = \frac{A\left(2x - 3\right) + B\left(x + 2\right)}{2x^2 + x - 6}$$

Now equate the numerators, we really no longer need to worry about the denominator, after-all they're the same. Basically we just need to find the constants that make these numerators equal. You should, of course, pick values for x that make solving for A and B easy. Let's see what values for x make our job easier.

$$4x - 13 = A\left(2x - 3\right) + B\left(x + 2\right)$$

I say we choose $x = -2$ to see what happens.

$$\begin{aligned}
4x - 13 &= A\left(2x - 3\right) + B\left(x + 2\right) \\
-8 - 13 &= A\left(-4 - 3\right) + B\left(-2 + 2\right) \\
-21 &= -7A \\
3 &= A
\end{aligned}$$

Okay, now that we know that $A = 3$, let's find B by choosing an easy value for x, let's try $x = 0$, but any value for x will do just fine.

$$\begin{aligned}
4x - 13 &= 3\left(2x - 3\right) + B\left(x + 2\right) \\
-13 &= 3\left(-3\right) + B\left(2\right) \\
-13 &= -9 + 2B \\
-4 &= 2B \\
-2 &= B
\end{aligned}$$

Now we finally have it.
$$\frac{4x - 13}{2x^2 + x - 6} = \frac{3}{x + 2} - \frac{2}{2x - 3}$$

Why bother though? Look at these equivalent integrals and see why!

$$\int \frac{4x - 13}{2x^2 + x - 6} \, dx = \int \frac{3}{x + 2} - \frac{2}{2x - 3} \, dx = \int \frac{3}{x + 2} \, dx - \int \frac{2}{2x - 3} \, dx$$

Of course we can now integrate.

$$\int \frac{3}{x + 2} \, dx - \int \frac{2}{2x - 3} \, dx = 3 \ln|x + 2| - \ln|2x - 3| + C$$

2. Let's return to the first example given, but this time I want to integrate.

$$
\begin{aligned}
\int \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} \, dx &= \int 2x + 3 \, dx + \int \frac{8x - 4}{3x^2 - 2x - 1} \, dx \\
&= \int 2x + 3 \, dx + \int \frac{A}{(3x + 1)} \, dx + \int \frac{B}{(x - 1)} \, dx
\end{aligned}
$$

Work: Now let's determine the constants A and B.

$$8x - 4 = A(x - 1) + B(3x + 1)$$

Let $x = 1$ and you'll get:
$$4 = 4B, \quad \Rightarrow \quad B = 1.$$

Now let $x = 0$ and you'll get:

$$-4 = -A + 1, \quad \Rightarrow \quad A = 5.$$

The we have:

$$
\begin{aligned}
\int \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} \, dx &= \int 2x + 3 \, dx + \int \frac{8x - 4}{3x^2 - 2x - 1} \, dx \\
&= \int 2x + 3 \, dx + \int \frac{5}{(3x + 1)} \, dx + \int \frac{1}{(x - 1)} \, dx
\end{aligned}
$$

Finally we need to do the simple integration:

$$\int 2x + 3 \, dx + \int \frac{5}{(3x + 1)} \, dx + \int \frac{1}{(x - 1)} \, dx = x^2 + 3x + \frac{5}{3} \ln|3x + 1| + \ln|x - 1| + C$$

3. We have, from page one, and a prior work sheet:

$$\int \frac{1}{1 - x^2} \, dx = \tanh^{-1} x + C.$$

Try to integrate this one using partial fraction decomposition.

Work:

$$\int \frac{1}{1-x^2}\,\mathrm{d}x \;=\; \int \frac{A}{1-x} + \frac{B}{1+x}\,\mathrm{d}x$$

$$=\; \int \frac{A}{1-x} + \frac{B}{1+x}\,\mathrm{d}x$$

$$=\; \int \frac{1/2}{1-x} + \frac{1/2}{1+x}\,\mathrm{d}x \qquad \text{We'll do this in class.}$$

$$=\; \frac{1}{2}\int \frac{1}{1-x}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{1+x}\,\mathrm{d}x$$

$$=\; -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C$$

$$=\; \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

You should not that we have troubles when $x = \pm 1$, just look at the original integrand. Furthermore if $x \in (-1, 1)$ we have

$$\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x$$
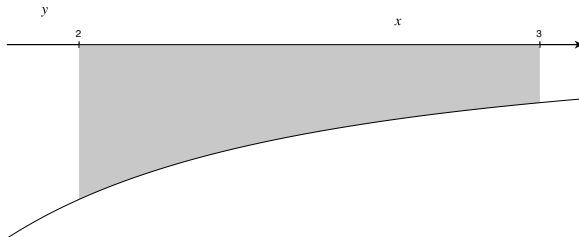
Let's take a look at a definite integral.

Figure 1: Integration of interest.

For example, this makes no sense because $x \notin (-1,\ 1)$:[3]

$$\int_2^3 \frac{1}{1-x^2}\,\mathrm{d}x = \tanh^{-1} x\big]_2^3 = \tanh^{-1} 3 - \tanh^{-1} 2 = \text{calculator may fail here,}$$

but this does:

$$\int_2^3 \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big]_2^3 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 3 = \ln\sqrt{\frac{2}{3}} \approx -0.2027$$

Take home message: start to use technology and have a good book (no, not the Bible) to consult.

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[3] Actually, each of these terms gives a complex number:

$$\tanh^{-1} 3 - \tanh^{-1} 2 = \left(\ln\sqrt{2} - \frac{\pi}{2}i\right) - \left(\ln\sqrt{3} - \frac{\pi}{2}i\right) = \ln\sqrt{\frac{2}{3}}$$

2.2 Examples

1. Evaluate $\displaystyle\int \frac{x^2 + 2}{(x-1)\,(2x-8)\,(x+2)}\,\mathrm{d}x$

2. Evaluate $\displaystyle\int \frac{x^3 + 1}{x^2 - 4}\,\mathrm{d}x$

3. Evaluate $\displaystyle\int \frac{3x - 9}{(x-1)\,(x+2)^2}\,\mathrm{d}x$

4. Evaluate $\displaystyle\int \frac{18}{(x+3)\,(x^2+9)}\,\mathrm{d}x$

5. Evaluate $\displaystyle\int \frac{4 - x}{x\,(x^2+2)^2}\,\mathrm{d}x$

6. Evaluate $\displaystyle\int \frac{1}{x^2 + 2x + 8}\,\mathrm{d}x$

7. Partial fractions, as you now know, can be quite laborious. So please consider using a computer algebra system to do some of the above problems. See next page for the Mathematica code for these problems.

3 Greek To Me!

Let's be thankful that we have a native *Greek* speakers (Yes, there here at ECC) to help us out when we get stuck.

Lower case greek letter used in mathematics. alpha $= \alpha$, beta $= \beta$, gamma $= \gamma$, delta $= \delta$, epsilon $= \epsilon$, var epsilon $= \varepsilon$, zeta $= \zeta$, eta $= \eta =$, theta $= \theta$, var theta $= \vartheta$, iota $= \iota$, kappa $= \kappa$, lambda $= \lambda$, mu $= \mu$, nu $= \nu$, xi $= \xi$, pi $= \pi$, var pi $= \varpi$, rho $= \rho$, var rho $= \varrho$, sigma $= \sigma$, var sigma $= \varsigma$, tau $= \tau$, upsilon $= \upsilon$, phi $= \phi$, var phi $= \varphi$, chi $= \chi$, psi $= \psi$, omega $= \omega$.

Upper case Greek letters used in mathematics: gamma $= \Gamma$, delta $= \Delta$, theta $= \Theta$, lambda $= \Lambda$, xi $= \Xi$, pi $= \Pi$, sigma $= \Sigma$, upsilon $= \Upsilon$, phi $= \Phi$, psi $= \Psi$, Omega $= \Omega$.

In[8]:= **Integrate[(x^2 + 2) / ((x - 1) (2 x - 8) (x + 2)), x]**

Out[8]= $\frac{1}{2} \left(\text{Log}[-4 + \text{x}] - \frac{1}{3} \text{Log}[-1 + \text{x}] + \frac{1}{3} \text{Log}[2 + \text{x}] \right)$

In[9]:= **Integrate[(x^3 + 1) / (x^2 - 4), x]**

Out[9]= $\frac{\text{x}^2}{2} + \frac{9}{4} \text{Log}[-2 + \text{x}] + \frac{7}{4} \text{Log}[2 + \text{x}]$

In[10]:= **Integrate[(3 x - 9) / ((x - 1) (x + 2)^2), x]**

Out[10]= $3 \left(-\frac{5}{3 (2 + \text{x})} - \frac{2}{9} \text{Log}[-1 + \text{x}] + \frac{2}{9} \text{Log}[2 + \text{x}] \right)$

In[11]:= **Integrate[(18) / ((x + 3) (x^2 + 9)), x]**

Out[11]= $18 \left(\frac{1}{18} \text{ArcTan}\left[\frac{\text{x}}{3} \right] + \frac{1}{18} \text{Log}[3 + \text{x}] - \frac{1}{36} \text{Log}[9 + \text{x}^2] \right)$

In[12]:= **Apart[(x^2 - 2) / ((x + 2) (x^2 + 4)^3)]**

Out[12]= $\frac{1}{256 (2 + \text{x})} + \frac{3 (-2 + \text{x})}{4 (4 + \text{x}^2)^3} + \frac{2 - \text{x}}{32 (4 + \text{x}^2)^2} + \frac{2 - \text{x}}{256 (4 + \text{x}^2)}$

In[13]:= **Apart[(x^2 + 2) / ((x - 1) (2 x - 8) (x + 2))]**

Out[13]= $\frac{1}{2 (-4 + \text{x})} - \frac{1}{6 (-1 + \text{x})} + \frac{1}{6 (2 + \text{x})}$

In[14]:= **Apart[(x^3 + 1) / (x^2 - 4)]**

Out[14]= $\frac{9}{4 (-2 + \text{x})} + \text{x} + \frac{7}{4 (2 + \text{x})}$

In[15]:= **Apart[(3 x - 9) / ((x - 1) (x + 2)^2)]**

Out[15]= $-\frac{2}{3 (-1 + \text{x})} + \frac{5}{(2 + \text{x})^2} + \frac{2}{3 (2 + \text{x})}$

In[16]:= **Apart[(18) / ((x + 3) (x^2 + 9))]**

Out[16]= $\frac{1}{3 + \text{x}} + \frac{3 - \text{x}}{9 + \text{x}^2}$

In[17]:= **Apart[(x^2 - 2) / ((x + 2) (x^2 + 4)^3)]**

Out[17]= $\frac{1}{256 (2 + \text{x})} + \frac{3 (-2 + \text{x})}{4 (4 + \text{x}^2)^3} + \frac{2 - \text{x}}{32 (4 + \text{x}^2)^2} + \frac{2 - \text{x}}{256 (4 + \text{x}^2)}$

In[18]:= **Apart[1 / (x^5 + 2 x + 2)]**

Out[18]= $\frac{1}{2 + 2 \text{x} + \text{x}^5}$

Figure 2: Mathematica Code