MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #5 — Sakai Web Project Material

1 Tough Problems?

1. Supposed you're asked to calculate the integral

$$\int \frac{1}{x^2+4x+13}\ \mathrm{d}x?$$

Many would just stop because they can't factor. Look over the prior notes and you may realize that this quadratic factor in the numerator is irreducible. So it is probably related to the only irreducible form you know:

$$\int \frac{1}{x^2+1}\ \mathrm{d}x.$$

Let's proceed to get it in that form.

$$\begin{aligned}
\int \frac{1}{x^2+4x+13}\ \mathrm{d}x &= \int \frac{1}{x^2+4x+4+9}\ \mathrm{d}x \\
&= \int \frac{1}{(x+2)^2+9}\ \mathrm{d}x \\
&= \frac{1}{9}\int \frac{1}{((x+2)/3)^2+1}\ \mathrm{d}x
\end{aligned}$$

Now let $u = (x+2)/3$, then $3\ \mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}
\frac{1}{9}\int \frac{1}{((x+2)/3)^2+1}\ \mathrm{d}x &= \frac{1}{3}\int^{*} \frac{1}{u^2+1}\ \mathrm{d}u \\
&= \frac{1}{3}\arctan u + C \\
&= \boxed{\frac{1}{3}\arctan\frac{x+2}{3} + C}
\end{aligned}$$

2. I've noticed that even a slight variation in the way a question is worded can prevent even the best students from following through. Let's say that you're asked to find the antiderivative of

$$\frac{x^2-x+2}{x^3-x^2+x-1}$$

that passes through the point $(2,\ 6)$. What would you do?

Well, here's what I'd do.

$$\int \frac{x^2-x+2}{x^3-x^2+x-1}\ \mathrm{d}x = \int \frac{x^2-x+2}{(x-1)(x^2+1)}\ \mathrm{d}x$$

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised September 8, 2009.

Certainly a partial fraction decomposition of the form:

$$\frac{x^2 - x + 2}{(x-1)\left(x^2+1\right)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}.$$

Using the method discussed in class, we have:

$$x^2 - x + 2 = A\left(x^2+1\right) + (Bx + C)(x-1).$$

Let $x = 1$ and we have:

$$2 = 2A \qquad \Rightarrow \qquad A = 1.$$

Now let $x = 0$ and we have:

$$2 = 1 - C \qquad \Rightarrow \qquad C = -1.$$

Now let $x = 2$ and we have:

$$4 = 5 + (2B - 1) \qquad \Rightarrow \qquad B = 0.$$

So now we can continue the integration.[2]

$$\begin{aligned} \int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1}\, \mathrm{d}x &= \int \frac{x^2 - x + 2}{(x-1)\left(x^2+1\right)}\, \mathrm{d}x \\ &= \int \frac{1}{x-1} - \frac{1}{x^2+1}\, \mathrm{d}x \\ &= \ln|x-1| - \arctan x + k \end{aligned}$$

The general antiderivative is

$$f(x) = \ln|x-1| - \arctan x + k,$$

and to find the constant k you'll need to use the point $(2,\ 6)$, *i.e.* $f(2) = 6$.

$$\begin{aligned} 6 &= \ln|1| - \arctan 2 + k \\ 6 + \arctan 2 &= k \end{aligned}$$

Finally we have:

$$\boxed{f(x) = \ln|x-1| - \arctan x + 6 + \arctan 2}$$

3. Okay, enough already! Not really, we could actually go on for days if not years. Very last one, today at least. Integrate!

$$\int \frac{e^x}{e^{2x} + e^x - 2}\, \mathrm{d}x$$

First let $u = e^x$, then $\mathrm{d}u = e^x\ \mathrm{d}x$.

$$\begin{aligned} \int \frac{e^x}{e^{2x} + e^x - 2}\, \mathrm{d}x &= \int^* \frac{1}{u^2 + u - 2}\, \mathrm{d}u \\ &= \int^* \frac{1}{(u-1)(u+2)}\, \mathrm{d}u \end{aligned}$$

[2] I am using k as the constant of integration and not the usual C, because I already used C in the work.

Now we need to figure out the partial fraction decomposition of the integrand. Here's the short version.

$$1 = A\,(u+2) + B\,(u-1)$$

Let $u = 1$ and we get $A = 1/3$; and if we let $u = -2$ and we get $B = -1/3$. Continuing the integration.

$$\begin{aligned}
\int^{*} \frac{1}{(u-1)\,(u+2)}\;\mathrm{d}u &= \frac{1}{3}\int^{*} \frac{1}{(u-1)} - \frac{1}{(u+2)}\;\mathrm{d}u \\
&= \frac{1}{3}\ln\left|\frac{u-1}{u+2}\right| + C \\
&= \boxed{\ln\sqrt[3]{\frac{|e^x-1|}{e^x+2}} + C}
\end{aligned}$$

2 Integration Overview

Here's a partial list of integrals that I think everyone should be familiar with, although I don't think there's universal agreement on this list, it's a good basis for moving forward. I'm placing a boxed star $\boxed{\star}$ next to those that you should *absolutely* know![3]

1. $\displaystyle\int x^n\;\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$ $\quad\boxed{\star^{100}}$
2. $\displaystyle\int \frac{1}{x}\;\mathrm{d}x = \ln|x| + C$ $\quad\boxed{\star^{50}}$
3. $\displaystyle\int e^x\;\mathrm{d}x = e^x + C$ $\quad\boxed{\star^{\infty}}$
4. $\displaystyle\int a^x\;\mathrm{d}x = \frac{a^x}{\ln a} + C$
5. $\displaystyle\int \sin x\;\mathrm{d}x = -\cos x + C$ $\quad\boxed{\star^{25}}$
6. $\displaystyle\int \cos x\;\mathrm{d}x = \sin x + C$ $\quad\boxed{\star^{25}}$
7. $\displaystyle\int \sec^2 x\;\mathrm{d}x = \tan x + C$
8. $\displaystyle\int \csc^2 x\;\mathrm{d}x = -\cot x + C$
9. $\displaystyle\int \sec x\;\tan x\;\mathrm{d}x = \sec x + C$
10. $\displaystyle\int \csc x\;\cot x\;\mathrm{d}x = -\csc x + C$
11. $\displaystyle\int \sec x\;\mathrm{d}x = \ln|\sec x + \tan x| + C$

[3] At least while taking courses related to calculus.

12. $\int \csc x \ \mathrm{d}x = \ln|\csc x - \cot x| + C$

13. $\int \tan x \ \mathrm{d}x = \ln|\sec x| + C$

14. $\int \cot x \ \mathrm{d}x = \ln|\sin x| + C$

15. $\int \sinh x \ \mathrm{d}x = \cosh x + C$ $\boxed{\star}$

16. $\int \cosh x \ \mathrm{d}x = \sinh x + C$ $\boxed{\star}$

17. $\int \frac{1}{1+x^2} \ \mathrm{d}x = \arctan x + C$ $\boxed{\star^3}$

18. $\int \frac{1}{\sqrt{1-x^2}} \ \mathrm{d}x = \arcsin x + C$ $\boxed{\star^2}$

Anyway, you're taking this class mainly to learn the various methods of integration. Certainly, u-substitution should be at the top of your list, then *integration by parts*, then for rational functions you'll need to try the method of *partial fractions*. These techniques will not work on all integrals though. In fact you'll hit some integrals that will befuddle the best among us—some, even more maddening, are impossible to find a closed form.[4] It is not trivial to integrate many functions, but you *luckily* live in an age where machines can help. Getting stuck is a fact of life, but please bear in mind that the book is giving you problems that work nicely, even if it seems like a lot of work.

The recipes that follow are not to be memorized, but bear in mind that many instructors require their students to memorize long lists of integrals, as well as recipes. However, I again want to emphasize that you should maintain a short list of basic integrals and a short list of basic techniques. However, you should be able to follow a recipe.

3 Trigonometric Integrals

Working problems can in fact lead to nice recipes, and here's a list related to trigonometric integrals of a particular form.

3.1 Integrals Involving Sine and Cosine

1. If the $\boxed{\text{power of the sine is odd and positive}}$, save one factor of sine and convert the remaining sines to cosines.

$$\int \sin^{2k+1} x \ \cos^n x \ \mathrm{d}x = \int \sin x \ \left(\sin^2 x\right)^k \ \cos^n x \ \mathrm{d}x = \int \left(1 - \cos^2 x\right)^k \ \cos^n x \ \sin x \ \mathrm{d}x$$

[4] Try this one on a computer algebra system:

$$\int e^{-x^2} \ \mathrm{d}x.$$

You'll see that most calculators will just return the integral unchanged. However, more sophisticated computer packages will actually return a constant multiple of a function called the *error function*, abbreviated $\mathsf{Erf}(x)$.

Now let $u = \cos x$, then $\mathrm{d}u = -\sin x \ \mathrm{d}x$.

Example:

$$\begin{aligned}
\int \sin^3 x \ \cos^4 x \ \mathrm{d}x &= \int \sin^2 x \ \cos^4 x \ \sin x \ \mathrm{d}x \\
&= \int \left(1 - \cos^2 x\right) \ \cos^4 x \ \sin x \ \mathrm{d}x \\
&= -\int^{*} \left(1 - u^2\right) \ u^4 \ \mathrm{d}u \\
&= \int^{*} u^6 - u^4 \ \mathrm{d}u \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \boxed{\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C}
\end{aligned}$$

2. If the $\boxed{\text{power of the cosine is odd and positive}}$, save one factor of cosine and convert the remaining cosines to sines.

$$\int \cos^{2k+1} x \ \sin^n x \ \mathrm{d}x = \int \cos x \ \left(\cos^2 x\right)^k \ \sin^n x \ \mathrm{d}x = \int \left(1 - \sin^2 x\right)^k \ \sin^n x \ \cos x \ \mathrm{d}x$$

Now let $u = \sin x$, then $\mathrm{d}u = \cos x \ \mathrm{d}x$.

Example:

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \ \mathrm{d}x &= \int_{\pi/6}^{\pi/3} \frac{\cos x \ \cos^2 x}{\sqrt{\sin x}} \ \mathrm{d}x \\
&= \int_{\pi/6}^{\pi/3} \frac{\cos x \ \left(1 - \sin^2 x\right)}{\sqrt{\sin x}} \ \mathrm{d}x \\
&= \int_{1/2}^{\sqrt{3}/2} \frac{\left(1 - u^2\right)}{\sqrt{u}} \ \mathrm{d}u \\
&= \int_{1/2}^{\sqrt{3}/2} \left(u^{-1/2} - u^{3/2}\right) \ \mathrm{d}u \\
&= \left(2u^{1/2} - \frac{2u^{5/2}}{5}\right)\Bigg]_{1/2}^{\sqrt{3}/2} \\
&= \frac{17\sqrt[4]{3} - 19}{10\sqrt{2}} \qquad \text{I cheated and used a computer.}
\end{aligned}$$

Okay, so the arithmetic is getting nasty here! Actually, I really cheated, and just did the entire integral on a computer to see if my work is okay. It's really important to learn how to use a computer to do mathematics.

Here's the code:

```
In[2]:= Integrate[(Cos[x])^3/Sqrt[Sin[x]], {x, π/6, π/3}]
```

$$\text{Out[2]=}\ -\frac{19}{10\sqrt{2}} + \frac{17\ 3^{1/4}}{10\sqrt{2}}$$

Figure 1: Mathematica Code

It appears to be pure magic and the real work that Mathematica does is completely transparent. Transparent work[5] is never good and everyone needs to labor through these problems, lest you think it's no work at all.

3. If both the powers of the cosine and sine is even and non-negative, make repeated use of these identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

to rewrite (expand) the integrand to odd powers of the cosines. Don't worry, I'll be reasonable here.

Example:

$$\begin{aligned}
\int \cos^4 x \, \mathrm{d}x &= \int \left(\cos^2 x\right)^2 \, \mathrm{d}x \\
&= \int \left(\frac{1+\cos 2x}{2}\right)^2 \, \mathrm{d}x \\
&= \int \frac{1+2\cos 2x+\cos^2 2x}{4} \, \mathrm{d}x \\
&= \int \frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4} \, \mathrm{d}x \\
&= \int \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos 4x}{8} \, \mathrm{d}x \\
&= \int \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{8} + \frac{\cos 4x}{8} \, \mathrm{d}x \\
&= \int \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8} \, \mathrm{d}x \\
&= \boxed{\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}
\end{aligned}$$

[5] Work in another's job and you'll tend to appreciate what they do.

Once again, here's the code:

```
In[1]:= Integrate[(Cos[x])^4, x]

Out[1]= 3 x/8 + 1/4 Sin[2 x] + 1/32 Sin[4 x]

In[2]:= D[%1, x]

Out[2]= 3/8 + 1/2 Cos[2 x] + 1/8 Cos[4 x]

In[4]:= Simplify[%2]

Out[4]= Cos[x]^4
```

Figure 2: Mathematica Code

3.2 Integrals Involving Tangent and Secant

1. If the $\boxed{\text{power of the secant is even and positive}}$, save one factor of $\sec^2 x$ and convert the remaining secants to tangents.

$$\begin{aligned}\int \sec^{2k} x \ \tan^n x \ \mathrm{d}x &= \int \sec^2 x \ \sec^{2k-2} x \ \tan^n x \ \mathrm{d}x \\ &= \int \sec^2 x \ \left(\sec^2 x\right)^{k-1} \ \tan^n x \ \mathrm{d}x \\ &= \int \sec^2 x \ \left(1+\tan^2 x\right)^{k-1} \ \tan^n x \ \mathrm{d}x\end{aligned}$$

Now let $u = \tan x$, then $\mathrm{d}u = \sec^2 x \ \mathrm{d}x$.

Example:

$$\begin{aligned}\int_0^{\pi/4} \sec^4 \theta \ \tan^4 \theta \ \mathrm{d}\theta &= \int_0^{\pi/4} \sec^2 \theta \ \tan^4 \theta \ \sec^2 \theta \ \mathrm{d}\theta \\ &= \int_0^{\pi/4} \left(1+\tan^2 \theta\right) \ \tan^4 \theta \ \sec^2 \theta \ \mathrm{d}\theta\end{aligned}$$

Now let $u = \tan x$, then $\mathrm{d}u = \sec^2 x \ \mathrm{d}x$.

$$\begin{aligned}\int_0^{\pi/4} \left(1+\tan^2 \theta\right) \ \tan^4 \theta \ \sec^2 \theta \ \mathrm{d}\theta &= \int_0^1 \left(1+u^2\right) \ u^4 \ \mathrm{d}u \\ &= \int_0^1 \left(u^4+u^6\right) \ \ \mathrm{d}u \\ &= \left.\frac{7u^5+5u^7}{35}\right]_0^1 \\ &= \boxed{\frac{12}{35}}\end{aligned}$$

Yes, Mathematica gives the same result. I checked!

2. If the $\boxed{\text{power of the tangent is odd and positive}}$, save one factor of $\sec x \ \tan x$ and convert the remaining tangents to secants.

$$\begin{aligned}
\int \sec^n x \ \tan^{2k+1} x \ \mathrm{d}x &= \int \sec^{n-1} x \ \tan^{2k} x \ \tan x \ \sec x \ \mathrm{d}x \\
&= \int \sec^{n-1} x \ \left(\tan^2 x\right)^k \ \tan x \ \sec x \ \mathrm{d}x \\
&= \int \sec^{n-1} x \ \left(\sec^2 x - 1\right)^k \ \tan x \ \sec x \ \mathrm{d}x
\end{aligned}$$

Now let $u = \sec x$, then $\mathrm{d}u = \sec x \ \tan x \ \mathrm{d}x$.

Example:

$$\begin{aligned}
\int_0^{\pi/3} \tan^5 x \ \sec^4 x \ \mathrm{d}x &= \int_0^{\pi/3} \tan^4 x \ \sec^3 x \ \tan x \ \sec x \ \mathrm{d}x \\
&= \int_0^{\pi/3} \left(\tan^2 x\right)^2 \ \sec^3 x \ \tan x \ \sec x \ \mathrm{d}x \\
&= \int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \ \sec^3 x \ \tan x \ \sec x \ \mathrm{d}x
\end{aligned}$$

Now let $u = \sec x$, then $\mathrm{d}u = \sec x \ \tan x \ \mathrm{d}x$.

$$\begin{aligned}
\int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \ \sec^3 x \ \tan x \ \sec x \ \mathrm{d}x &= \int_1^2 \left(u^2 - 1\right)^2 \ u^3 \ \mathrm{d}u \\
&= \int_1^2 \left(u^4 - 2u^2 + 1\right) \ u^3 \ \mathrm{d}u \\
&= \int_1^2 \left(u^7 - 2u^5 + u^3\right) \ \mathrm{d}u \\
&= \left.\frac{3u^8 - 8u^6 - 6u^4}{24}\right]_1^2 \\
&= \boxed{\frac{117}{8}}
\end{aligned}$$

Yes, Mathematica gives the same result. I checked!

4 Examples

1. Show.[6]
$$\int \tan x \ \mathrm{d}x = \ln|\sec x| + C$$

2. Show.[7]
$$\int \sec x \ \mathrm{d}x = \ln|\sec x + \tan x| + C$$

3. Show.[8]
$$\int \csc x \ \mathrm{d}x = \ln|\csc x - \cot x| + C$$

4. Integrate.[9]
$$\int \frac{\tan^2 x}{\cos x} \ \mathrm{d}x$$

5. Integrate.[10]
$$\int \frac{\cos x + \sin x}{\sin 2x} \ \mathrm{d}x$$

6. Integrate.
$$\int \frac{1}{\cos x - 1} \ \mathrm{d}x$$

Again, even with hints, you might be lost. However, I want to stress that practice will make these problems seem trivial. The reason for giving hints is to push everyone in the same *reasonable* direction. You should, if resourceful, be able to follow your own lead. Bear in mind, hints are given to guide the practiced mind.

[6] Hint: rewrite tangent as a ratio, and then use u-substitution. We did this before, but I think many will not recall this. Although nice to know, it is not required knowledge.

[7] Hint: rewrite integrand, multiplying it by the unit factor
$$1 = \frac{\sec x + \tan x}{\sec x + \tan x},$$
then u-substitution, with $u = \sec x + \tan x$.

[8] Hint: rewrite integrand, multiplying it by the unit factor
$$1 = \frac{\csc x - \cot x}{\csc x - \cot x},$$
then u-substitution, with $u = \csc x - \cot x$.

[9] Hint: rewrite in terms of tangents and secants. You'll need to use problem 2, and you'll need to use integration by parts.

[10] Hint: it's related to problem 2 and 3.