MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #7 — Sakai Web Project Material

1 Trigonometric Substitutions, §7.3

In this section of the textbook we will be presented with difficult integrands that contain a function of x that looks like:

$$\boxed{\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}}.$$

In general we will be looking to make a substitution, but in these cases our substitution will be another function with another parameter. To make our calculations simpler, we need to make these *new* functions invertible, that is, they should be *one-to-one*. For example, here we're using a new invertible function $g(\theta)$ to rewrite the integrand $f(x)$:

$$\int f(x)\ \mathrm{d}x = \int f(g(\theta))\ g'(\theta)\ \mathrm{d}\theta,$$

which really looks like classic u-substitution, and it may even be referred to as *inverse u-substitution* by some in the *mathematical digerati*. So let's take a look at some examples, and I will divide it into three cases.

1. The integrand contains an expression of the form $\sqrt{a^2 - x^2}$, just use $x = a\sin\theta$. Now as you already know, the sine function is not invertible, so we need to restrict the domain of $\theta \in [-\pi/2,\ \pi/2]$, and will use $1 - \sin^2\theta = \cos^2\theta$ to help simplify the integral. It should be noted that in general, $\sqrt{\cos^2\theta} = |\cos\theta|$, but for the restricted domain of theta, we have $\sqrt{\cos^2\theta} = \cos\theta$.

   Example:

$$\int \sqrt{1 - 9x^2}\ \mathrm{d}x = 3\int \sqrt{1/9 - x^2}\ \mathrm{d}x$$

   Now let $x = \dfrac{\sin\theta}{3}$, then $3\ \mathrm{d}x = \cos\theta\ \mathrm{d}\theta$.

$$\begin{aligned} 3\int \sqrt{1/9 - x^2}\ \mathrm{d}x &= \int^* \sqrt{\frac{1 - \sin^2\theta}{9}}\ \cos\theta\ \mathrm{d}\theta \\ &= \frac{1}{3}\int^* \sqrt{1 - \sin^2\theta}\ \cos\theta\ \mathrm{d}\theta \\ &= \frac{1}{3}\int^* \sqrt{\cos^2\theta}\ \cos\theta\ \mathrm{d}\theta \\ &= \frac{1}{3}\int^* \cos\theta\ \cos\theta\ \mathrm{d}\theta \\ &= \frac{1}{3}\int^* \cos^2\theta\ \mathrm{d}\theta \end{aligned}$$

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised January 10, 2009.

Recalling the methods from the prior worksheet, we're going to make use of,

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2},$$

to continue the process.

$$\begin{aligned}\frac{1}{3}\int^{*}\cos^2\theta\ \mathrm{d}\theta &= \frac{1}{3}\int^{*}\frac{\cos 2\theta + 1}{2}\ \mathrm{d}\theta \\ &= \frac{1}{3}\left(\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right)+C \\ &= \frac{\sin 2\theta}{12}+\frac{\theta}{6}+C\end{aligned}$$

Okay, I really hate to say it, but we need to return to original variable, x. Yikes![2]

$$\int\sqrt{1-9x^2}\ \mathrm{d}x = \boxed{\frac{x\sqrt{1-9x^2}}{2}+\frac{\arcsin 3x}{6}+C}$$

Give Mathematica a try and see how quickly it does this integral, and it is in full agreement with our result!

2. The integrand contains an expression of the form $\sqrt{a^2+x^2}$, just use $x = a\tan\theta$. Now as you already know, the tangent function is not invertible, so we need to restrict the domain of $\theta \in (-\pi/2,\ \pi/2)$, and will use $1+\tan^2\theta = \sec^2\theta$ to help simplify the integral. It should be noted that in general, $\sqrt{\sec^2\theta} = |\sec\theta|$, but for the restricted domain of theta, we have $\sqrt{\sec^2\theta} = \sec\theta$.

Example:

$$\int\frac{1}{x^2\sqrt{x^2+4}}\ \mathrm{d}x$$

Now let $x = 2\tan\theta$, then $\mathrm{d}x = 2\sec^2\theta\ \mathrm{d}\theta$.

$$\begin{aligned}\int\frac{1}{x^2\sqrt{x^2+4}}\ \mathrm{d}x &= \int^{*}\frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\ 2\sec^2\theta\ \mathrm{d}\theta \\ &= \int^{*}\frac{2\sec^2\theta}{8\tan^2\theta\sec\theta}\ \mathrm{d}\theta \\ &= \frac{1}{4}\int^{*}\frac{\sec\theta}{\tan^2\theta}\ \mathrm{d}\theta \\ &= \frac{1}{4}\int^{*}\frac{\cos\theta}{\sin^2\theta}\ \mathrm{d}\theta\end{aligned}$$

Oh, another substitution is in order. This time we will let $u = \sin\theta$, then $\mathrm{d}u = \cos\theta\ \mathrm{d}\theta$.

$$\begin{aligned}\frac{1}{4}\int^{*}\frac{\cos\theta}{\sin^2\theta}\ \mathrm{d}\theta &= \frac{1}{4}\int^{**}\frac{1}{u^2}\ \mathrm{d}u \\ &= \frac{1}{4}\int^{**}u^{-2}\ \mathrm{d}u \\ &= -\frac{1}{4u}+C \\ &= -\frac{1}{4\sin\theta}+C\end{aligned}$$

[2] I'll do the trigonometry in class, it's really not *too* bad.

Again, I really hate to say it, but we need to return to original variable, x. Yikes![3]

$$\int \frac{1}{x^2\sqrt{x^2+4}}\ \mathrm{d}x = \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

Give Mathematica a try and see how quickly it does this integral, and it is in full agreement with our result!

3. The integrand contains an expression of the form $\sqrt{x^2-a^2}$, just use $x = a\sec\theta$. Now as you already know, the secant function is not invertible, so we need to restrict the domain of $\theta \in [0,\ \pi/2) \cup [\pi,\ 3\pi/2)$, and will use $\tan^2\theta = \sec^2\theta - 1$ to help simplify the integral. It should be noted that in general, $\sqrt{\tan^2\theta} = |\tan\theta|$, but for the restricted domain of theta, we have $\sqrt{\tan^2\theta} = \tan\theta$.

   Example:

$$\int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}}\ \mathrm{d}x$$

   Now let $3x = \sec\theta$, then $3\ \mathrm{d}x = \sec\theta\tan\theta\ \mathrm{d}\theta$.

$$\begin{aligned}
\int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}}\ \mathrm{d}x &= \int_{\pi/4}^{\pi/3} \frac{3^4}{\sec^5\theta\sqrt{\sec^2\theta-1}}\ \sec\theta\tan\theta\ \mathrm{d}\theta \\
&= 81\int_{\pi/4}^{\pi/3} \cos^4\theta\ \mathrm{d}\theta
\end{aligned}$$

   Once again, recalling the methods from the prior worksheet, we're going make use of,

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2},$$

   to continue the process.

$$\begin{aligned}
81\int_{\pi/4}^{\pi/3} \cos^4\theta\ \mathrm{d}\theta &= 81\int_{\pi/4}^{\pi/3} \left(\frac{\cos 2\theta + 1}{2}\right)^2\ \mathrm{d}\theta \\
&= 81\int_{\pi/4}^{\pi/3} \frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4}\ \mathrm{d}\theta \\
&= 81\int_{\pi/4}^{\pi/3} \frac{\cos^2 2\theta}{4} + \frac{\cos 2\theta}{2} + \frac{1}{4}\ \mathrm{d}\theta \\
&= 81\int_{\pi/4}^{\pi/3} \frac{\cos 4\theta + 1}{8} + \frac{\cos 2\theta}{2} + \frac{1}{4}\ \mathrm{d}\theta \\
&= 81\int_{\pi/4}^{\pi/3} \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\ \mathrm{d}\theta
\end{aligned}$$

   Okay, now it can be evlauated!

$$\begin{aligned}
81\int_{\pi/4}^{\pi/3} \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\ \mathrm{d}\theta &= 81\left(\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3\theta}{8}\right)\Bigg]_{\pi/4}^{\pi/3} \\
&= \boxed{\frac{567\sqrt{3}}{64} + \frac{81\pi}{32} - \frac{81}{4}}
\end{aligned}$$

[3] I'll do the trigonometry in class, it's really not *too* bad.

1.1 Examples

1. Integrate. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,\mathrm{d}x$

2. Integrate.[4] $\displaystyle\int \sqrt{4x^2+20}\,\mathrm{d}x$

3. Integrate. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,\mathrm{d}x$

[4]You may find useful, a reduction formula:

$$\int \sec^m x\ \mathrm{d}x = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\ \mathrm{d}x.$$

1.2 Solutions

1. Integrate. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\ \mathrm{d}x$

   Work: Let $x = 2\sin\theta$, then $\mathrm{d}x = 2\cos\theta\ \mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{x^2}{(4-x^2)^{3/2}}\ \mathrm{d}x &= \int \frac{x^2}{\left(\sqrt{4-x^2}\right)^3}\ \mathrm{d}x \\
&= \int^* \frac{4\sin^2\theta}{\left(\sqrt{4-4\sin^2\theta}\right)^3}\ 2\cos\theta\ \mathrm{d}\theta \\
&= \int^* \frac{4\sin^2\theta}{(2\cos\theta)^3}\ 2\cos\theta\ \mathrm{d}\theta \\
&= \int^* \frac{4\sin^2\theta}{8\cos^3\theta}\ 2\cos\theta\ \mathrm{d}\theta \\
&= \int^* \frac{8\sin^2\theta}{8\cos^2\theta}\ \mathrm{d}\theta \\
&= \int^* \tan^2\theta\ \mathrm{d}\theta
\end{aligned}$$

   Now what? Looks like we're stuck? No, not yet!

$$\begin{aligned}
\int^* \tan^2\theta\ \mathrm{d}\theta &= \int \sec^2\theta - 1\ \mathrm{d}\theta \\
&= \tan\theta - \theta + C
\end{aligned}$$

   Now, of course, we will need to return the original variable x.

$$\int \frac{x^2}{(4-x^2)^{3/2}}\ \mathrm{d}x = \boxed{\frac{x}{\sqrt{4-x^2}} - \arcsin\frac{x}{2} + C}$$

   Mathematica produces the same result.

2. Integrate. $\displaystyle\int \sqrt{4x^2+20}\ \mathrm{d}x$

   Work: Let's start. We're letting $x = \sqrt{5}\tan\theta$, then $\mathrm{d}x = \sqrt{5}\sec^2\theta\ \mathrm{d}\theta$

$$\begin{aligned}
\int \sqrt{4x^2+20}\ \mathrm{d}x &= 2\int \sqrt{x^2+5}\ \mathrm{d}x \\
&= 2\int^* \sqrt{5\tan^2\theta+5}\ \sqrt{5}\sec^2\theta\ \mathrm{d}\theta \\
&= 2\int^* \sqrt{5\sec^2\theta}\ \sqrt{5}\sec^2\theta\ \mathrm{d}\theta \\
&= 10\int^* \sec^3\theta\ \mathrm{d}\theta
\end{aligned}$$

   Now using the reduction formula with $m = 3$ we get:

$$10\int^* \sec^3\theta\ \mathrm{d}\theta = 10\left(\frac{\tan\theta\sec\theta}{2} + \frac{1}{2}\int^* \sec\theta\ \mathrm{d}\theta\right) = 5\tan\theta\sec\theta + 5\int^* \sec\theta\ \mathrm{d}\theta.$$

I hope you at least *recognize* the remaining integral.

$$5\tan\theta\sec\theta + 5\ln|\sec\theta + \tan\theta| + C.$$

Now we must return to the original variable.

$$\int \sqrt{4x^2+20}\ \mathrm{d}x = \boxed{x\sqrt{x^2+5} + 5\ln\left|\frac{\sqrt{x^2+5}+x}{\sqrt{5}}\right| + C}$$

3. Integrate. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\ \mathrm{d}x$

Work: Here I am using $x = 3\sec\theta$, then $\mathrm{d}x = 3\sec\theta\tan\theta\ \mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-9}}\ \mathrm{d}x &= \int^* \frac{1}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}\ 3\sec\theta\tan\theta\ \mathrm{d}\theta \\
&= \frac{1}{9}\int^* \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\ \mathrm{d}\theta \\
&= \frac{1}{9}\int^* \cos\theta\ \mathrm{d}\theta \\
&= \frac{1}{9}\sin\theta + C
\end{aligned}$$

Returning to the original variable.

$$\int \frac{1}{x^2\sqrt{x^2-9}}\ \mathrm{d}x = \boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$