

1 Numerical Integration

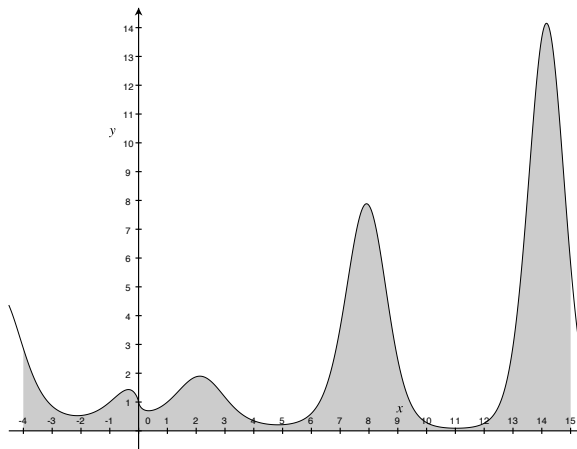


Figure 1: Area of interest.

Suppose we're asked to evaluate

$$\int_{-4}^{15} |x|^{\sin x} dx ?$$

I think almost everyone would be stumped. However, in reality it is not always necessary to find a definitive answer, but instead an approximation may suffice and in fact may be preferred. Approximations can be hard to come by, but today's mathematical tools are quite good. For your own sake, you should try to do this problem with the tools you have at hand: calculator, desktop computer software, or even a computer language such as FORTRAN or C.

Tools are nice to have, and here I am using an application called Grapher that is quite capable of doing numerical integration. Here's what it give me using three different built-in methods:

- Romberg Method: 44.8113
- Euler Method: 44.8114
- Runge Kutta Fourth Order Method: 44.8115

Mathematica actually choked² on it, but then quickly *burped* out 44.8114, which is the average of Grapher's values anyway.

¹This document was prepared by Ron Bannon (ron.bannon@mathography.org) using L^AT_EX 2_ε. Last revised January 10, 2009.

²Here's what it reported before giving me the value, "NIntegrate failed to converge to prescribed accuracy after 7 recursive bisections in x near $x = -0.06640625$."

As some of you may already know, I almost never write down a decimal answer, but sometimes I have no choice. The reason I have no choice here is that I cannot do this integration, so I need to use a method of approximation. Here's the Mathematica code. You should also notice that

```
In[20]:= Plot[Abs[x]^(Sin[x]), {x, -4, 15}]

Out[20]= - Graphics -

In[21]:= Integrate[Abs[x]^(Sin[x]), {x, -4, 15}]

Out[21]=  $\int_{-4}^{15} \text{Abs}[x]^{\text{Sin}[x]} dx$ 

In[22]:= NIntegrate[Abs[x]^(Sin[x]), {x, -4, 15}]

NIntegrate::ncvb : NIntegrate failed to converge to
prescribed accuracy after 7 recursive bisections in x near x = -0.0664063. More...

Out[22]= 44.8114
```

Figure 2: Mathematica Code

Mathematica cannot do the integration either and I have to resort to a new command to force Mathematica to use a numerical method. Numerical methods are approximate, so although we might conjecture about an exact value, it is doubtful that we will be able to find a closed form of this integral.

Numerical integration is used on definite integrals when we cannot find the antiderivative, or where speed is essential and approximations suffice. Being a numerical technique, we will rarely find the exact value of the definite integral. The techniques discussed in this worksheet are easily implemented on a computer algebra system, however Mathematica actually has some fairly robust numerical methods built in, with loads of options.³

1.1 Midpoint Rule

Given a definite integral

$$\int_a^b f(x) dx,$$

the n^{th} midpoint approximation⁴ is:

$$\frac{b-a}{n} [f(c_1) + f(c_2) + \cdots + f(c_n)],$$

where

$$c_i = a + (2i - 1) \frac{b-a}{2n}.$$

³Check out §17.2.2 of Heikki Ruskeepää's textbook, *Mathematica Navigator* (second edition), for more information—yes, the ECC library has a copy.

⁴This will be discussed in class.

Example: Using $n = 6$ compute the midpoint approximation of

$$\int_1^4 \sqrt{x} \, dx = \left. \frac{2x\sqrt{x}}{3} \right|_1^4 = \frac{14}{3} = 4\frac{2}{3} = 4.\bar{6}$$

and compare your approximate answer to the exact answer.

Work:

$$\int_1^4 \sqrt{x} \, dx \approx 0.5 \left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75} \right] \approx 4.66925$$

Here's the Mathematica code for this problem.

```
In[11]:= Integrate[Sqrt[x], {x, 1, 4}]
Out[11]= 14/3
In[12]:= N[%11, 7]
Out[12]= 4.666667
In[13]:= (4 - 1) / 6 * Sum[Sqrt[1 + (2 i - 1) (4 - 1) / (2 * 6)], {i, 1, 6}]
Out[13]= 1/2 (3/2 + sqrt(5)/2 + sqrt(7)/2 + sqrt(11)/2 + sqrt(13)/2 + sqrt(15)/2)
In[16]:= N[%13, 7]
Out[16]= 4.669245
In[15]:= NIntegrate[Sqrt[x], {x, 1, 4}]
Out[15]= 4.66667
```

Figure 3: Mathematica Code

1.2 Trapezoid Rule

Given a definite integral

$$\int_a^b f(x) \, dx,$$

the n^{th} trapezoidal approximation⁵ is:

$$\frac{b-a}{2n} [f(a) + 2f(c_1) + \cdots + 2f(c_{n-1}) + f(b)],$$

where

$$c_i = a + \frac{(b-a)i}{n}.$$

Example: Using $n = 6$ compute the trapezoidal approximation of

$$\int_1^4 \sqrt{x} \, dx = \left. \frac{2x\sqrt{x}}{3} \right|_1^4 = \frac{14}{3} = 4\frac{2}{3} = 4.\bar{6}$$

⁵This will be discussed in class.

and compare your approximate answer to the exact answer.

Work:

$$\int_1^4 \sqrt{x} \, dx \approx 0.25 \left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4} \right] \approx 4.66925$$

Here's the Mathematica code for this problem.

```
In[11]:= Integrate[Sqrt[x], {x, 1, 4}]
Out[11]= 14/3

In[12]:= N[%11, 7]
Out[12]= 4.666667

In[13]:= (4 - 1) / 6 * Sum[Sqrt[1 + (2 i - 1) (4 - 1) / (2 * 6)], {i, 1, 6}]
Out[13]= 1/2 (3/2 + sqrt(5)/2 + sqrt(7)/2 + sqrt(11)/2 + sqrt(13)/2 + sqrt(15)/2)

In[16]:= N[%13, 7]
Out[16]= 4.669245

In[15]:= NIntegrate[Sqrt[x], {x, 1, 4}]
Out[15]= 4.66667
```

Figure 4: Mathematica Code

1.3 Simpson's Rule

Here, n needs to be even. Given a definite integral

$$\int_a^b f(x) \, dx,$$

the n^{th} Simpson's Rule approximation⁶ is:

$$\frac{b-a}{3n} [f(a) + 4f(c_1) + 2f(c_2) + \cdots + 2f(c_{n-2}) + 4f(c_{n-1}) + f(b)],$$

where

$$c_i = a + \frac{b-a}{n}i.$$

Example: Using $n = 6$ compute the Simpson's Rule approximation of

$$\int_1^4 \sqrt{x} \, dx = \left. \frac{2x\sqrt{x}}{3} \right|_1^4 = \frac{14}{3} = 4\frac{2}{3} = 4.\bar{6}$$

and compare your approximate answer to the exact answer.

Work:

$$\int_1^4 \sqrt{x} \, dx \approx \frac{1}{6} \left[1 + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + 2 \right] \approx 4.66656$$

⁶This will be discussed in class.

1.4 Calculator and Mathematica Examples

If you're looking through your textbook, you'll notice that they give error bounds for the various methods discussed here. However, I really just want you to keep in mind that numerical methods are not exact, and when you're doing the homework, I suggest that you use a computer algebra system and consult the textbook for error bounds.

1. Approximate the definite integral

$$\int_0^2 \sin x^2 \, dx$$

- (a) using the Midpoint Rule with $n = 8$;
- (b) using the Trapezoid Rule with $n = 8$;
- (c) using Simpson's Rule with $n = 8$.

2. Finding $t \leq \pi^2$.

$$\int_0^t \sin \sqrt{x} \, dx = 2$$

- (a) Look at the graph to estimate t . Aspect ratio is not one-to-one, but the shaded region is 2 square units.

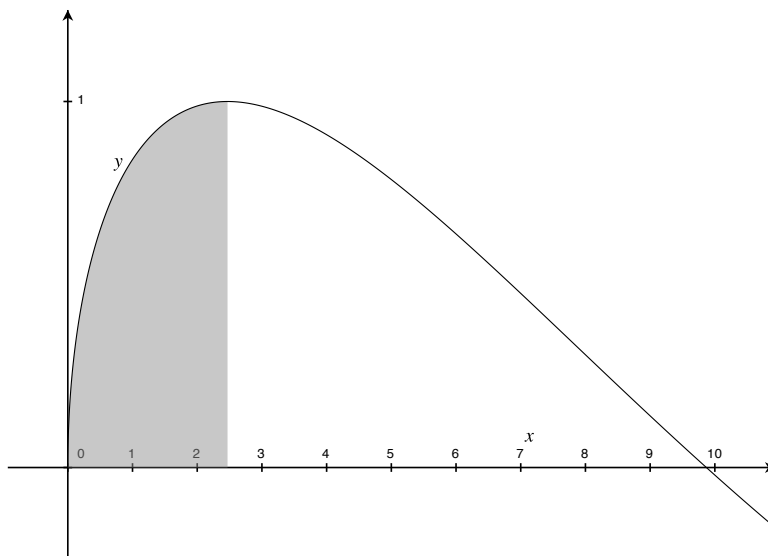


Figure 5: The shaded region is approximately 2 square units.

- (b) Use your guess and the Mathematica code, `NIntegrate[Sin[Sqrt[x]], {x, 0, YOUR GUESS}]`, to refine t to at least four decimal places.
- (c) Show.

$$\int \sin \sqrt{x} \, dx = 2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C$$

(d) Find the exact value of t and compare it with your approximate for t .

3. The *normal distribution* is commonly used in statistics. The standard normal ($\mu = 0$ and $\sigma = 1$) probability distribution function (pdf) is:

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

and computing

$$\int_a^b f(x) \, dx$$

for $a < b \in \mathbb{R}$ is necessary. However, this integral is very difficult to deal with, so extensive tables have been constructed to facilitate evaluation of this integral.

(a) Graph $f(x)$.

(b) The area under f . $\int_{-\infty}^{\infty} f(x) \, dx$.

(c) The area to the right of the mean. $\int_0^{\infty} f(x) \, dx$.

(d) The area to the left of the mean. $\int_{-\infty}^0 f(x) \, dx$.

(e) The area within one sigma of the mean. $\int_{-1}^1 f(x) \, dx$.

(f) The area within two sigmas of the mean. $\int_{-2}^2 f(x) \, dx$.

(g) The area within three sigmas of the mean. $\int_{-3}^3 f(x) \, dx$.

1.5 Solutions to Calculator and Mathematica Examples

1. Approximate the definite integral

$$\int_0^2 \sin x^2 \, dx$$

- (a) using the Midpoint Rule with $n = 8$;

Work: Mathematica evaluates this integral, and returns 0.804776.

$$\begin{aligned} \int_0^2 \sin x^2 \, dx &\approx 0.25 [\sin 0.125^2 + \sin 0.375^2 + \sin 0.625^2 + \sin 0.875^2 \\ &\quad + \sin 1.125^2 + \sin 1.375^2 + \sin 1.625^2 + \sin 1.875^2] \\ &\approx \boxed{0.811882} \end{aligned}$$

- (b) using the Trapezoid Rule with $n = 8$;

Work: Mathematica evaluates this integral, and returns 0.804776.

$$\begin{aligned} \int_0^2 \sin x^2 \, dx &\approx 0.125 [\sin 0^2 + 2 \sin 0.25^2 + 2 \sin 0.5^2 + 2 \sin 0.75^2 \\ &\quad + 2 \sin 1^2 + 2 \sin 1.25^2 + 2 \sin 1.5^2 + 2 \sin 1.75^2 + \sin 2^2] \\ &\approx \boxed{0.790821} \end{aligned}$$

- (c) using Simpson's Rule with $n = 8$.

Work: Mathematica evaluates this integral, and returns 0.804776.

$$\begin{aligned} \int_0^2 \sin x^2 \, dx &\approx \frac{1}{12} [\sin 0^2 + 4 \sin 0.25^2 + 2 \sin 0.5^2 + 4 \sin 0.75^2 \\ &\quad + 2 \sin 1^2 + 4 \sin 1.25^2 + 2 \sin 1.5^2 + 4 \sin 1.75^2 + \sin 2^2] \\ &\approx \boxed{0.806337} \end{aligned}$$

I'd like to point out that Mathematica can actually evaluate this integral, but the exact form may be a bit strange. Anyway, we're interested in approximating this, so please use numerical integration.

2. Finding $t \leq \pi^2$.

$$\int_0^t \sin \sqrt{x} \, dx = 2$$

- (a) Look at the graph to estimate t . Aspect ratio is not one-to-one, but the shaded region is 2 square units.

Work: Answers may vary, but it clearly is between 2 and 3, and I estimate it to be $\boxed{2.4}$. No one should go beyond one decimal place, and everyone should be near 2.5.

- (b) Use your guess and the Mathematica code, `NIntegrate[Sin[Sqrt[x]], x, 0, YOUR GUESS]`, to refine t to at least four decimal places.

Work: `NIntegrate[Sin[Sqrt[x]], {x, 0, 2.4674}]` returns 2.⁷

- (c) Show.

$$\int \sin \sqrt{x} \, dx = 2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C$$

Work: You can try integrating if you like, but here I am differentiating.

$$\begin{aligned} \frac{d}{dx} [2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C] &= \frac{d}{dx} [2 \sin \sqrt{x}] - \frac{d}{dx} [2\sqrt{x} \cos \sqrt{x}] + \frac{d}{dx} [C] \\ &= \frac{\cos \sqrt{x}}{\sqrt{x}} + \sin \sqrt{x} - \frac{\cos \sqrt{x}}{\sqrt{x}} \\ &= \sin \sqrt{x} \end{aligned}$$

Q.E.D.

- (d) Find the exact value of t and compare it with your approximate for t .

Work: Yes, I know this looks impossible, but it is not. In fact it can be solved by hand quite easily. However, just take a look at your approximate answer and this fact.

$$\begin{aligned} \int_0^t \sin \sqrt{x} \, dx &= [2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x}]_0^t \\ 2 &= 2 \sin \sqrt{t} - 2\sqrt{t} \cos \sqrt{t} \\ t &= \boxed{\frac{\pi^2}{4}} \end{aligned}$$

3. The standard normal ($\mu = 0$ and $\sigma = 1$) probability distribution function (pdf) is:

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

⁷Actually the exact answer is $\pi^2/4$ which is approximately 2.467401100272339654708622749969 and I hope no one goes beyond that! Oh, Mathematica can take this decimal expansion to an enormous number of places. It's scary!

(a) Graph $f(x)$.

Work: This is the famous bell-shaped standard normal curve.

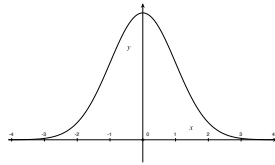


Figure 6: Partial graph of $f(x)$.

(b) The area under f .
$$\int_{-\infty}^{\infty} f(x) dx.$$

Work: Using Mathematica,

```
NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, -Infinity, Infinity}]
```

returns 1. That is, 100%.

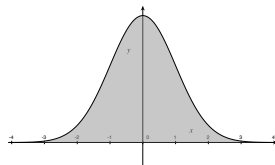


Figure 7: Partial graph of $f(x)$ with entire area shaded.

(c) The area to the right of the mean.
$$\int_0^{\infty} f(x) dx.$$

Work: Using Mathematica,

```
NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, 0, Infinity}]
```

returns 0.5. That is, 50% is above the mean.

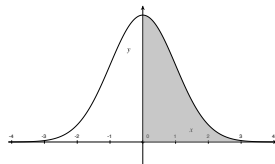


Figure 8: Partial graph of $f(x)$ with area to the right of the mean shaded.

- (d) The area to the left of the mean. $\int_{-\infty}^0 f(x) dx$.

Work: Using Mathematica,

`NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, -Infinity, 0}]`
returns `0.5`. That is, 50% is below the mean.

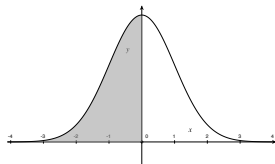


Figure 9: Partial graph of $f(x)$ with area to the left of the mean shaded.

- (e) The area within one sigma of the mean. $\int_{-1}^1 f(x) dx$.

Work: Using Mathematica,

`NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, -1, 1}]`
returns `0.6827`. That is, 68% is within one standard deviation of the mean.

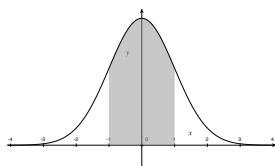


Figure 10: Partial graph of $f(x)$ with area within one sigma of the mean shaded.

- (f) The area within two sigmas of the mean. $\int_{-2}^2 f(x) dx$.

Work: Using Mathematica,

`NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, -2, 2}]`
returns `0.9545`. That is, 95% is within two standard deviation of the mean.

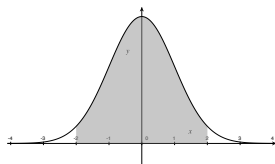


Figure 11: Partial graph of $f(x)$ with area within two sigmas of the mean shaded.

(g) The area within three sigmas of the mean. $\int_{-3}^3 f(x) dx$.

Work: Using Mathematica,

```
NIntegrate[1/Sqrt[2*Pi]*Exp[-x*x/2],{x, -3, 3}]
```

returns 0.9973. That is, nearly 100% is within three standard deviation of the mean.

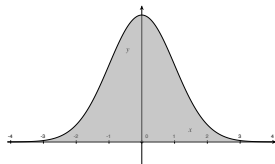


Figure 12: Partial graph of $f(x)$ with area within three sigmas of the mean shaded.