MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #9 — Sakai Web Project Material

1 Improper Integrals—Type I

You may recall that on the last handout we did the following three integrations—where f is a standard normal curve—using Mathematica's built-in numerical integration techniques.

- $\displaystyle\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1$
- $\displaystyle\int_{-\infty}^{0} f(x)\ \mathrm{d}x = \frac{1}{2}$
- $\displaystyle\int_{0}^{\infty} f(x)\ \mathrm{d}x = \frac{1}{2}$

These values here are indeed exact, but that is not the main issue of this particular handout. The feature of these integrals that is remarkable, is the actual limits, because all your prior limits have been finite.

Definition of an Improper Integral, Type I

1. If
$$\int_a^b f(x)\ \mathrm{d}x$$
exists for every $b \geq a$, then
$$\int_a^{\infty} f(x)\ \mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\ \mathrm{d}x$$
provided the limit exists.[2] If the limit exists we say its is *convergent*, if the limit does not exists we say it is *divergent*.

2. Similarly, if
$$\int_a^b f(x)\ \mathrm{d}x$$
exists for every $b \geq a$, then
$$\int_{-\infty}^{b} f(x)\ \mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\ \mathrm{d}x$$
provided the limit exists.[3] If the limit exists we say its is *convergent*, if the limit does not exists we say it is *divergent*.

[1]This document was prepared by Ron Bannon (ron.bannon@mathography.org) using $\LaTeX 2_\varepsilon$. Last revised January 10, 2009.

[2]Finite number

[3]Finite number

3. If we have the following

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x,$$

we split the integral into two parts by selecting any $a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{a} f(x)\ \mathrm{d}x + \int_{a}^{\infty} f(x)\ \mathrm{d}x$$

and *both* right-hand integrations must converge.

1.1 Example

1. Evaluate.

$$\int_{0}^{\infty} xe^{-x}\ \mathrm{d}x$$

Work: First change the infinite limit to some arbitrary finite limit. Here I'm using a, but you could use any letter except x in this case.[4]

$$\int_{0}^{\infty} xe^{-x}\ \mathrm{d}x = 1 - \frac{1}{e^a} - \frac{a}{e^a}$$

Now take the limit as $a \to \infty$.[5]

$$\int_{0}^{\infty} xe^{-x}\ \mathrm{d}x = \lim_{a\to\infty} \int_{0}^{a} xe^{-x}\ \mathrm{d}x = \lim_{a\to\infty} \left(1 - \frac{1}{e^a} - \frac{a}{e^a}\right) = 1$$

2 Improper Integrals—Type II

Definition of an Improper Integral, Type II

1. If f is continuous on $[a,\ b)$ and is discontinuous at b, then

$$\int_{a}^{b} f(x)\ \mathrm{d}x = \lim_{t\to b^-} \int_{a}^{t} f(x)\ \mathrm{d}x$$

if the limit exists.[6] If the limit exists we say its is *convergent*, if the limit does not exists we say it is *divergent*.

2. If f is continuous on $(a,\ b]$ and is discontinuous at a, then

$$\int_{a}^{b} f(x)\ \mathrm{d}x = \lim_{t\to a^+} \int_{t}^{b} f(x)\ \mathrm{d}x$$

if the limit exists.[7] If the limit exists we say its is *convergent*, if the limit does not exists we say it is *divergent*.

[4] I'm using integration by parts and this will be reviewed in class.
[5] This limit will be reviewed in class.
[6] Finite number
[7] Finite number

3. If f has a discontinuity at c, where $a < c < b$, and both

$$\int_a^c f(x)\ \mathrm{d}x \qquad \text{and} \qquad \int_c^b f(x)\ \mathrm{d}x$$

are convergent, then we define

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x$$

2.1 Example

1. Evaluate.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\ \mathrm{d}x$$

Work: The integral is improper because the integrand has a vertical asymptote at $x = 2$.

$$\begin{aligned}
\int_2^5 \frac{1}{\sqrt{x-2}}\ \mathrm{d}x &= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\ \mathrm{d}x \\
&= \lim_{t\to 2^+} 2\sqrt{x-2}\Big]_t^5 \\
&= \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) \\
&= \boxed{2\sqrt{3}}
\end{aligned}$$

3 Comparison Theorems

Comparison Theorem: Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\ \mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\ \mathrm{d}x$ is convergent.
2. If $\int_a^\infty g(x)\ \mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\ \mathrm{d}x$ is divergent.

It is useful to note that we often use the following integral for comparison purposes:

$$\int_1^\infty \frac{1}{x^p}\ \mathrm{d}x.$$

Here, it can be shown that this integral is convergent if $p > 1$ and divergent if $p \leq 1$.

3.1 Example

1. Is the integral
$$\int_1^\infty \frac{1+e^{-x}}{x}\ \mathrm{d}x$$
convergent or divergent?

   Work: It should be clear that
$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$
and we know that
$$\int_1^\infty \frac{1}{x}\ \mathrm{d}x$$
is divergent, so our integral
$$\int_1^\infty \frac{1+e^{-x}}{x}\ \mathrm{d}x$$
is $\boxed{\text{divergent}}$ by using the comparison theorem.

4 Examples

1. Show that
$$\int_1^\infty \frac{1}{x}\ \mathrm{d}x$$
is divergent.

2. Show
$$\int_0^\infty \frac{1}{1+x^2}\ \mathrm{d}x = \frac{\pi}{2}$$

3. Show

$$\int_0^\infty \frac{1}{e^x}\ \mathrm{d}x = 1$$

4. Show

$$\int_1^\infty \frac{1-x}{e^x}\ \mathrm{d}x = -\frac{1}{e}$$

5. Show

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\ \mathrm{d}x = \frac{\pi}{2}$$

6. Show that

$$\int_0^2 \frac{1}{x^3}\ \mathrm{d}x$$

is divergent.

7. Show

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\ \mathrm{d}x = \pi.$$

8. The solid formed by revolving the region between $1/x$ and the x-axis, for $x \geq 1$ is called Gabriel's Horn. What is most weird about this object is that it can be shown to have finite volume, but its surface area is infinite. Set-up and evaluate the integral for its volume and verify that it is π.

9. Show that

$$\int_0^\infty \frac{1}{e^{x^2}}\ \mathrm{d}x$$

is convergent.[8]

[8] **Hint:** notice that $e^{-x^2} \leq e^{-x}$, look over the past problems, and use the comparison theorem.