MTH 122 — Calculus II
Essex County College — Division of Mathematics and Physics[1]
Lecture Notes #10 — Sakai Web Project Material

1 Arc Length

Everyone should be familiar with the distance formula that was introduced in elementary algebra. It is a basic formula for the linear distance between two points in the plane. It states that the distance between $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This distance, of course, is for a line connecting those two points. However, what if we have a curve and we want to know the distance along that curve between two points? We will basically cut the curve into an infinite number of small linear sections, and then add these sections together to get the arc length, or distance between two points on the curve. Here a definite integral can be used to find the arc length, where we have a curve, $f(x)$, and two points on this curve that are connected by a curve that is continuously differentiable on the interval.

Arc Length Formula: If f' is continuous on $[a,\ b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ \mathrm{d}x.$$

This can also be written as

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

You should notice, that we're doing these integrations with respect to x, but you may recall from earlier problems, that it is sometimes easier to integrate with respect to y.

Arc Length Formula: If g' is continuous on $[c,\ d]$, then the length of the curve $x = g(y)$, $c \le y \le d$, is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\ \mathrm{d}y.$$

This can also be written as

$$L = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

Each of these formulas will be discussed in class, and I will mainly try to relate what we know about local linearity and integration to derive these formulas. The book, of course, proves these formulas and you may be interested in reading the textbook for a more precise explanation.

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised January 10, 2009.

1.1 Example

1. Set up an integral to compute the length of the curve $y = x^3$ from $x = 0$ to $x = 5$.

   Work: Here I will be using this formula.

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

   Plugging in, I get.

$$\begin{aligned} L &= \int_0^5 \sqrt{1 + (3x^2)^2}\ \mathrm{d}x \\ &= \int_0^5 \sqrt{1 + (3x^2)^2}\ \mathrm{d}x \\ &= \boxed{\int_0^5 \sqrt{1 + 9x^4}\ \mathrm{d}x} \end{aligned}$$

   You should notice that you were not asked to evaluate this integral.[2] Mathematica can actually integrate this, but it's confusing in its exact form.

2 Area of a Surface of Revolution

Definition: If the graph of a continuous function is revolved about a line, the resulting surface is a *surface of revolution*.

Here we will let $y = f(x)$, where f has a continuous derivative on the interval $[a,\ b]$. The area S of the *surface of revolution* formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2}\ \mathrm{d}x. \qquad \text{Here } y \text{ is a function of } x.$$

where r is the distance between the graph of f and the axis of revolution.

On the other hand, if $x = g(y)$ on the interval $[c,\ d]$, then the surface area is

$$S = 2\pi \int_a^b r(y) \sqrt{1 + [g'(x)]^2}\ \mathrm{d}y. \qquad \text{Here } x \text{ is a function of } y.$$

where r is the distance between the graph of g and the axis of revolution.

Each of these formulas will be discussed in class, and I will mainly try to relate what we know about local linearity and integration to derive these formulas. The book, of course, proves these formulas and you may be interested in reading the textbook for a more precise explanation.

[2] Approximately 125.68, which is slightly longer than the linear distance between those points on $y = x^3$.

2.1 Example

1. Find the area of the surface formed by revolving the graph of

$$f(x) = x^3$$

on the interval $[0,\ 1]$ about the x-axis.

Work: Here I am using the formula

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2}\ \mathrm{d}x.$$

Making the substitutions, I get.

$$\begin{aligned} S &= 2\pi \int_0^1 x^3 \sqrt{1 + [3x^2]^2}\ \mathrm{d}x \\ &= 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\ \mathrm{d}x \end{aligned}$$

Using simple u-substitution, it follows.

$$\begin{aligned} 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\ \mathrm{d}x &= \frac{\pi}{18} \int_1^{10} u^{1/2}\ \mathrm{d}u \\ &= \left.\frac{\pi u \sqrt{u}}{27}\right]_1^{10} \\ &= \boxed{\frac{\pi\left(10\sqrt{10} - 1\right)}{27}} \approx 3.563 \end{aligned}$$

3 Examples

1. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the $x - axis$.[3]

[3] This is not easy, but I think you'll eventually get

$$S = \pi\left[e\sqrt{1 + e^2} + \ln\left(e + \sqrt{1 + e^2}\right) - \sqrt{2} = \ln\left(1 + \sqrt{2}\right)\right].$$

2. Show that the length of arc of the graph

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval $[1/2,\ 2]$ is $33/16$.

3. Show that the length of arc of the graph

$$(y-1)^3 = x^2$$

on the interval $[0,\ 8]$ is $\left(40^{3/2} - 4^{3/2}\right)/27$.

4. Show that the surface formed by revolving

$$y = x^2$$

on the interval $\left[0,\ \sqrt{2}\right]$ about the y-axis is $13\pi/3$.

5. The solid formed by revolving the region between $1/x$ and the x-axis, for $x \geq 1$ is called Gabriel's Horn. What is most weird about this object is that it can be shown to have finite volume, but its surface area is infinite. Set-up and evaluate the integral for its surface area and show that this integral is divergent.

6. Find the total length of the graph of the astoid

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = 4.$$

A graph is provided as a guide.

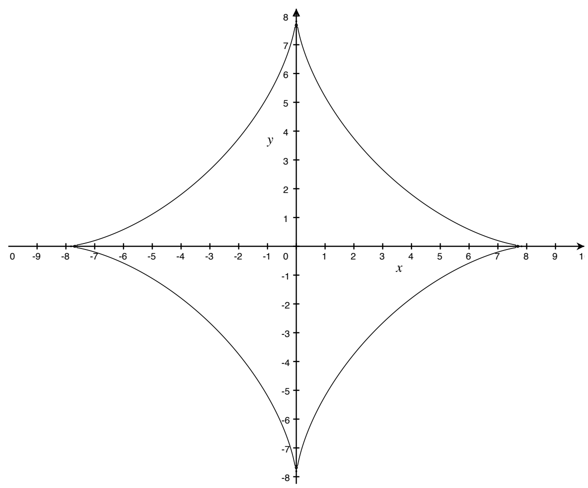

Figure 1: Complete graph of $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 4$.