MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #11 — Sakai Web Project Material

1 Approximating Functions with Polynomials

This particular sheet is based on an extra credit problem that I gave my MTH-121 students. It is based on taking derivatives and then fitting points to a particular polynomial, even though the differentiated function was not a polynomial. The basis of the project was to use increasing degree polynomials to approximate functions that have no obvious relationship to polynomials. For example the sine function near the origin can be approximated with an ever increasing degree polynomial of the form:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Let's take a look, and I'd like you to label the curves directly on the graph. Yes, they're related to the sine function and the expansion above. One curve is the sine function, then a sequence of polynomials of degree 1, 3, 5, 7, 9 and 11. Take a look, it's really amazing to see how this sequence proceeds.

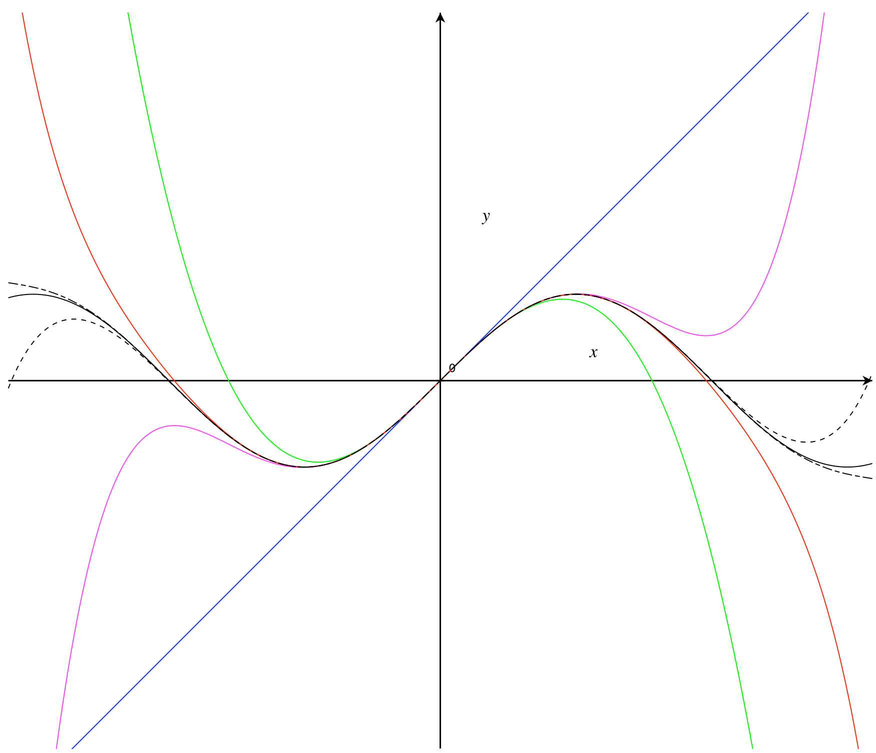

Figure 1: Sine and a sequence of polynomial functions. Seven graphs in all.

[1]This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised January 10, 2009.

1.1 Polynomial Approximation of Cosine

Although not rigorous, let's assume that the identity above for sine is correct. That is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots .$$

Furthermore, let's assume that the rules of differentiation apply to this infinite expansion. Let's see if we can generate the cosine from this.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots
\end{aligned}$$

Okay, it does seem reasonable, but let's now proceed with actually fitting a sequence of polynomials to the cosine function. The tangent line (degree one polynomial) approximation $L(x)$ is the best linear approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure the approximation is a good one, we stipulate the following:

$$\begin{aligned}
P(a) &= f(a) && P \text{ and } f \text{ should have the same value at } a. \\
P'(a) &= f'(a) && P' \text{ and } f' \text{ should have the same value at } a. \\
P''(a) &= f''(a) && P'' \text{ and } f'' \text{ should have the same value at } a.
\end{aligned}$$

1. Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies the above three conditions with $a = 0$. Graph P and f on the same axis. Does P fit f better than a tangent line in the region where $a = 0$?

   Work and Solution: We are given $f(x) = \cos x$ and $P(x) = A + Bx + Cx^2$, and we need their derivatives.

   $$\begin{aligned}
   f(x) &= \cos x \\
   f'(x) &= -\sin x \\
   f''(x) &= -\cos x \\
   P(x) &= A + Bx + Cx^2 \\
   P'(x) &= B + 2Cx \\
   P''(x) &= 2C
   \end{aligned}$$

   Now, using $a = 0$ we can determine the constants A, B and C. In the functions first.

   $$f(0) = \cos 0 = 1 = P(0) = A \qquad \Rightarrow \qquad A = 1$$

   Now in the first derivative.

   $$f'(1) = -\sin 0 = 0 = P'(0) = B \qquad \Rightarrow \qquad B = 0$$

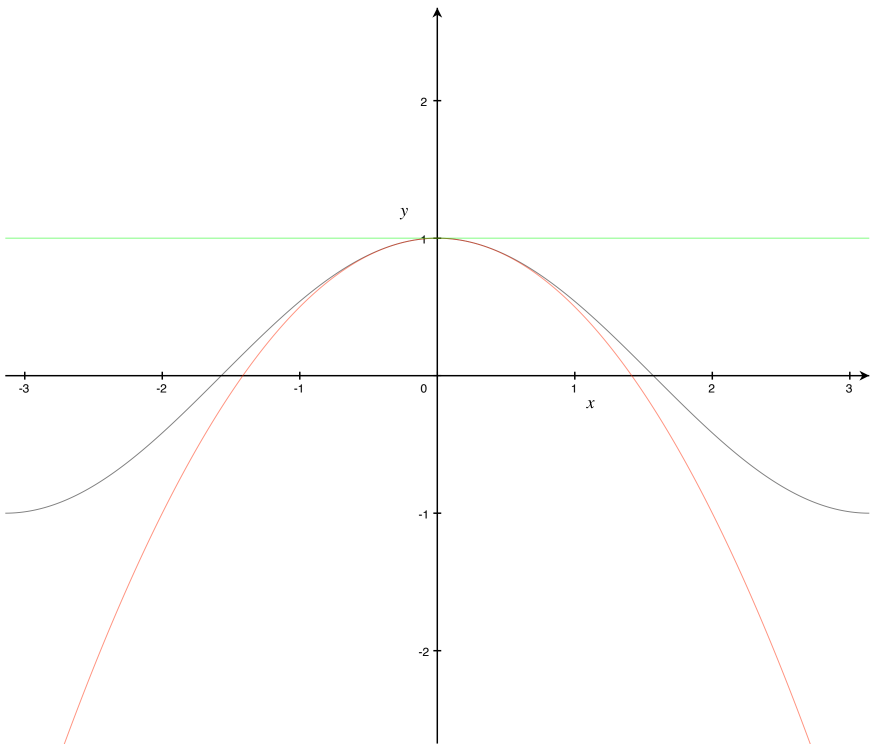

Figure 2: $P(x)$ in red, tangent in green, and $f(x)$ in gray.

Now in the second derivative.

$$f''(0) = -\cos 0 = -1 = P''(0) = 2C \qquad \Rightarrow \qquad C = -\frac{1}{2}$$

So, the function $\boxed{P(x) = 1 - x^2/2}$. Here's the graph.

$\boxed{\text{Yes}}$, the quadratic is a better fit than is the linear function.

2. If we repeat this method for higher and higher-degree polynomials, we find that $f(x)$ can be better approximated. Now repeat this method until you get a forth degree polynomial, $P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, that approximates $f(x)$ when $a = 0$. As before, graph P and f on the same axis. Does the forth degree P fit f better second degree P in the region where $a = 0$?

   Work and Solution: We are given $f(x) = \cos x$ and $P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, and we need their derivatives.

$$
\begin{aligned}
f(x) &= \cos x \\
f'(x) &= -\sin x \\
f''(x) &= -\cos x \\
f'''(x) &= \sin x \\
f^{(4)}(x) &= \cos x \\
P(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 \\
P'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 \\
P''(x) &= 2C + 6Dx + 12Ex^2 \\
P'''(x) &= 6D + 24Ex \\
P^{(4)}(x) &= 24E
\end{aligned}
$$

Now, using $a = 0$ we can determine the constants A, B, C, D and E. In the functions first.

$$f(0) = \cos 0 = 1 = P(0) = A \qquad \Rightarrow \qquad A = 1$$

Now in the first derivative.

$$f'(1) = -\sin 0 = 0 = P'(0) = B \qquad \Rightarrow \qquad B = 0$$

Now in the second derivative.

$$f''(0) = -\cos 0 = -1 = P''(0) = 2C \qquad \Rightarrow \qquad C = -\frac{1}{2}$$

Now in the third derivative.

$$f'''(0) = \sin 0 = 0 = P'''(0) = 6D \qquad \Rightarrow \qquad D = 0$$

Now in the forth derivative.

$$f^{(4)} = \cos 0 = 1 = P^{(4)}(0) = 24E \qquad \Rightarrow \qquad E = \frac{1}{24}$$

So, the function $\boxed{P(x) = 1 - x^2/2 + x^4/24}$.

Here's the graph.

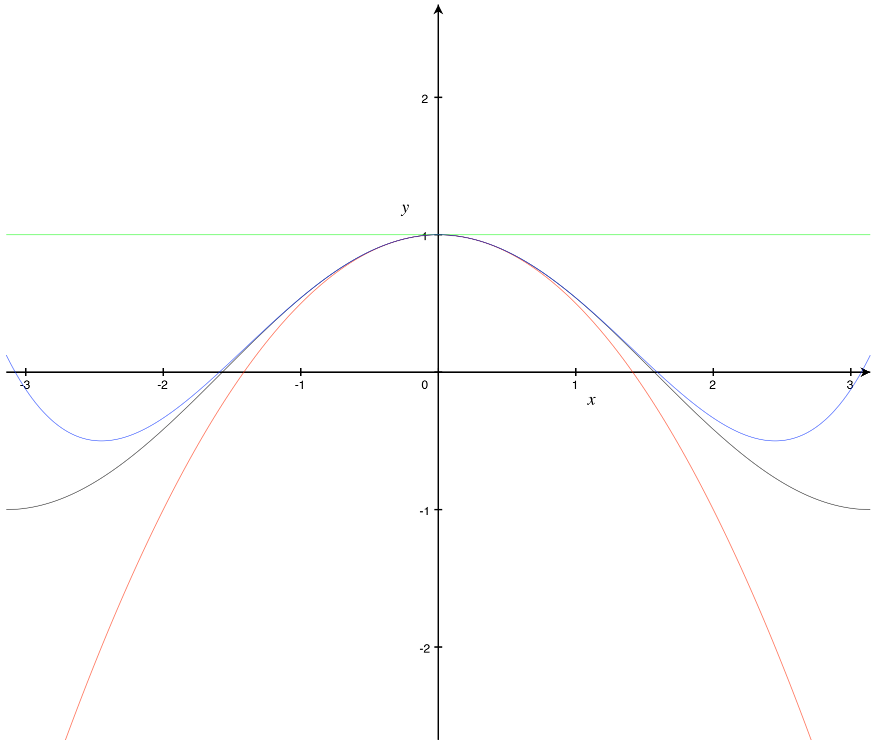

Figure 3: $1 - x^2/2 + x^4/24$ in blue, $1 - x^2/2$ in red, tangent in green, and $f(x)$ in gray.

$\boxed{\text{Yes}}$, the forth degree polynomial is a better fit than is the second degree polynomial.

This, of course can be repeated *ad infinitum*. For example the tenth degree polynomial approximation to $f(x)$ is

$$P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800}.$$

Here's the graph. Which, on this interval, is indistinguishable from the cosine curve.

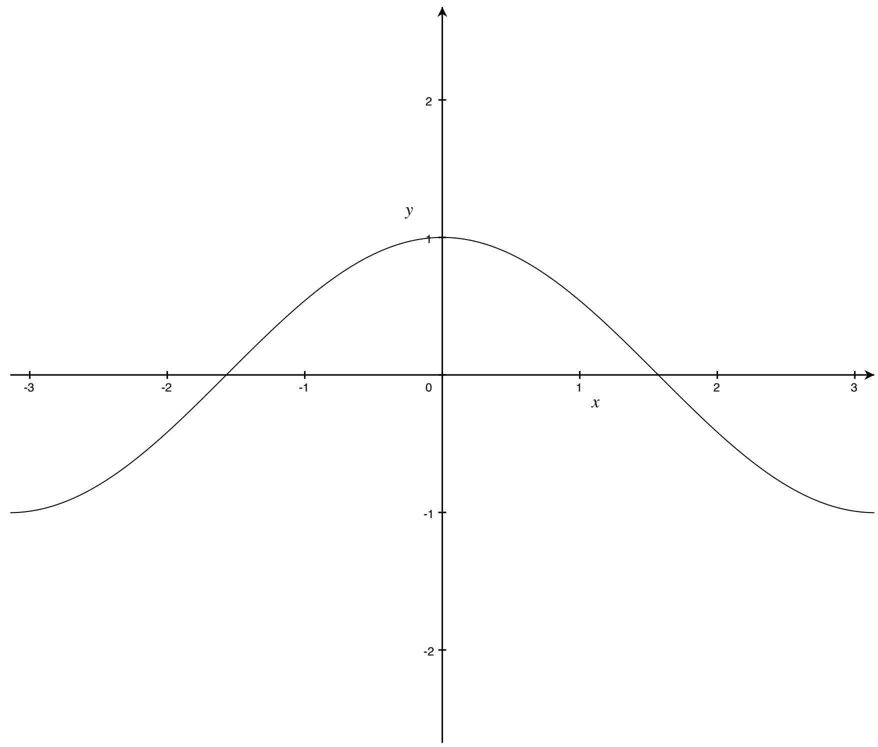

Figure 4: Looks just like the cosine curve.