MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #12 — Sakai Web Project Material

1 Taylor Polynomials

In the last class we actually generated several Taylor[2] polynomials and everyone should be clear that our example followed a pattern, as follows:

Degree 1: The Taylor Polynomial of degree 1 approximating $f(x)$ for x near zero is:

$$f(x) \approx f(0) + f'(0)\,x.$$

Degree 2: The Taylor Polynomial of degree 2 approximating $f(x)$ for x near zero is:

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2.$$

Certainly if we continue this process an easy pattern emerges. For example if we have an n^{th} degree polynomial of the form,

$$f(x) \approx C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots + C_{n-1}x^{n-1} + C_9 x^n,$$

and we follow the same process outlined in the prior worksheet, we'll get:

$$\begin{aligned} C_0 &= f(0) \\ C_1 &= \frac{f'(0)}{1!} \\ C_2 &= \frac{f''(0)}{2!} \\ C_3 &= \frac{f'''(0)}{3!} \\ C_4 &= \frac{f^{(4)}(0)}{4!} \\ \vdots &= \vdots \\ C_n &= \frac{f^{(n)}(0)}{n!} \end{aligned}$$

Finally we have a Taylor polynomial of degree n approximating $f(x)$ for x near 0 is,

$$\boxed{f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.}$$

[1]This document was prepared by Ron Bannon (ron.bannon@mathography.org) using LaTeX 2$_\varepsilon$. Last revised January 10, 2009.

[2]Brooke Taylor was an English Mathematician (1685–1731), but these approximating polynomials were known prior to Taylor's exposition on the subject.

1.1 Examples

1. Find the Taylor polynomial of degree 9 about $x = 0$ for the function $f(x) = e^x$.

   Work: This one is pretty easy, mainly because the derivative of e^x never changes. So we have:

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}}.$$

   Here's a graph of both $f(x) = e^x$ (in black) and the ninth degree polynomial (in red). You should notice that the fit is not perfect, but it looks damn good for $x > -3$. Although not obvious, the higher degree for this polynomial the better the fit becomes.

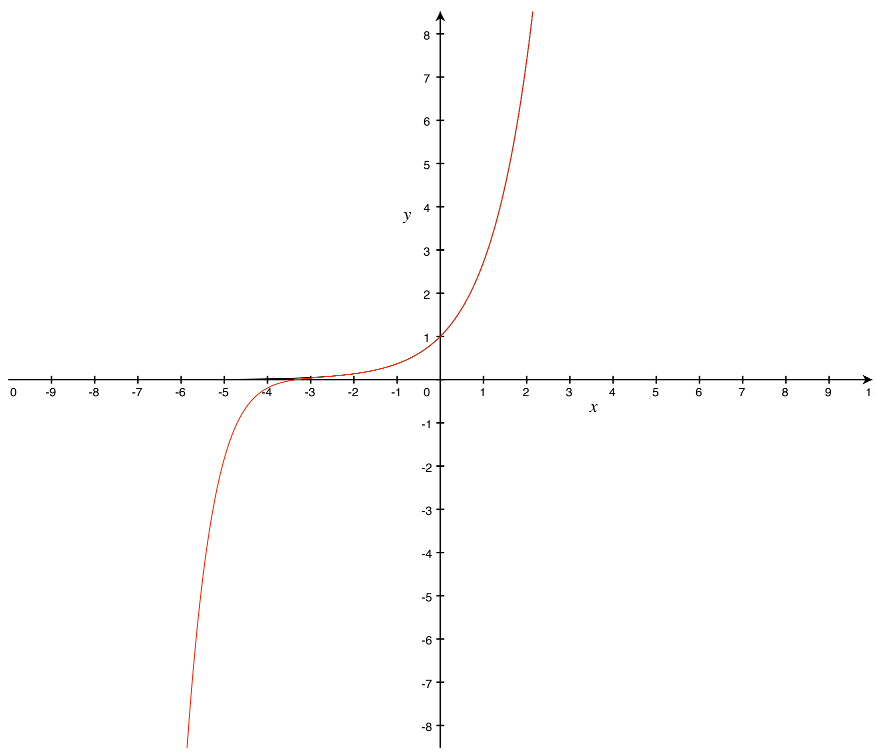

Figure 1: Looking good for $x > -3$.

2. Find the Taylor polynomial of degree 9 about $x = 0$ for the function $f(x) = \sin x$.

   Work: This one is also pretty easy, mainly because the derivative of $\sin x$, evaluated at $x = 0$, follows a nice sequence, $\{1,\ 0,\ -1,\ 0,\ 1,\ \ldots\}$. So we have:

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}}.$$

   Here's a graph of both $f(x) = \sin x$ (in black) and the ninth degree polynomial (in red). You should notice that the fit is not perfect, but it looks damn good for $-3.5 < x < 3.5$. Although not obvious, the higher degree for this polynomial the better the fit becomes.

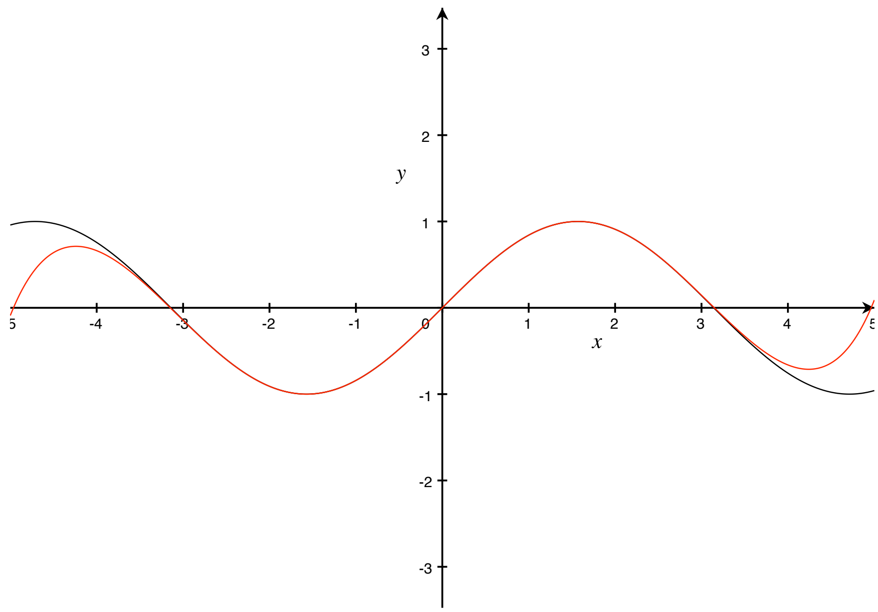

Figure 2: Looking good for $-3.5 < x < 3.5$.

3. Find the Taylor polynomial of degree 15 about $x = 0$ for the function $f(x) = \dfrac{1}{1-x}$.

Work: The derivative here is a bit more difficult. Let's look at what happens.

$$\begin{aligned}
f(x) &= (1-x)^{-1} \\
f'(x) &= 1 \cdot (1-x)^{-2} \\
f''(x) &= 1 \cdot 2\,(1-x)^{-3} \\
f'''(x) &= 1 \cdot 2 \cdot 3 \cdot (1-x)^{-4} \\
f^{(4)}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot (1-x)^{-5} \\
f^{(5)}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (1-x)^{-6} \\
\vdots &= \vdots \\
f^{(15)}(x) &= 15! \cdot (1-x)^{-16}
\end{aligned}$$

Now, let's evaluate each of these derivatives at $x = 0$.

$$\begin{aligned}
f'(0) &= 1! \\
f''(0) &= 2! \\
f'''(0) &= 3! \\
f^{(4)}(0) &= 4! \\
f^{(5)}(0) &= 5! \\
\vdots &= \vdots \\
f^{(15)}(0) &= 15!
\end{aligned}$$

So we have (after a bit of simplification):

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^{13} + x^{14} + x^{15}}$$

Here's a graph of both $f(x) = (1-x)^{-1}$ (in black) and the fifteenth degree polynomial (in red). You should notice that the fit is not perfect, but it looks damn good for $-1 < x < 1$. Although not obvious, the higher degree for this polynomial the better the fit becomes, but only for $x \in (-1,\ 1)$.

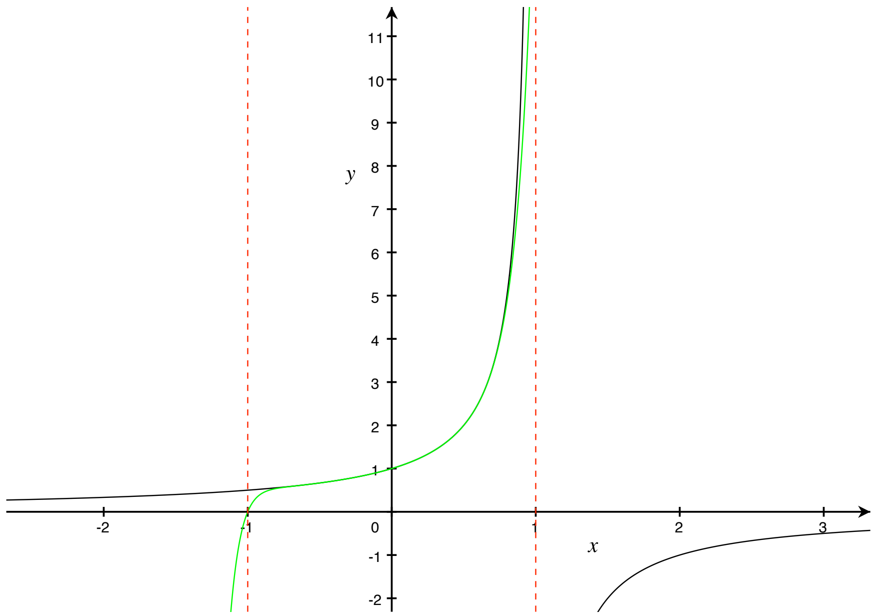

Figure 3: Looking good for $-1 < x < 1$.

You may recall that the infinite geometric sum is of the form:

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots + x^{13} + x^{14} + x^{15} + \cdots$$

and is convergent (*i.e.* works) for $x \in (-1,\ 1)$ only.[3]

Again, I must emphasize that unlike the other two examples, this particular Taylor polynomial is only valid for $-1 < x < 1$ no matter the degree.

[3]Geometric sums were covered in precalculus. But again, this is not about memorization, and I can only hope that it is at least recognized when shown.

2 A List of Important Taylor Series

You should notice that I am calling them series, and the main reason why is that equally is only true for the infinite expansion. Also, these equalities may not be true for all x, restrictions are indicated to the right.

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots & -1 < x < 1 \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & -1 \le x \le 1
\end{aligned}$$

3 Examples

1. Find the Taylor polynomial of degree 9 about $x = 0$ for the function $f(x) = \ln(1 + x)$.

2. Find the Taylor series for the function $f(x) = \ln(1 + x)$.

3. Use a graphing utility to graph the a sequence of Taylor polynomials and see if you can guess the interval of convergence.

4. Find the Taylor polynomial about $x = 0$ for the function $f(x) = (1 + x)^3$.

5. Use what you already know to *intuitively* show that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots.$$

6. Use what you already know to *intuitively* show that

$$x^2 \cos x^2 = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \cdots.$$

3.1 Solutions

1. Find the Taylor polynomial of degree 9 about $x = 0$ for the function $f(x) = \ln(1 + x)$.

   Work: Taking derivatives.

$$\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= (1+x)^{-1} \\ f''(x) &= -(1+x)^{-2} \\ f'''(x) &= 2 \cdot (1+x)^{-3} \\ f^{(4)}(x) &= -3! \cdot (1+x)^{-4} \\ f^{(5)}(x) &= 4! \cdot (1+x)^{-5} \\ \vdots &= \vdots \end{aligned}$$

   Now, let's evaluate each of these derivatives at $x = 0$.

$$\begin{aligned} f'(0) &= 1 \\ f''(0) &= -1 \\ f'''(0) &= 2! \\ f^{(4)}(0) &= -3! \\ f^{(5)}(0) &= 4! \\ \vdots &= \vdots \end{aligned}$$

   So we have (after a bit of simplification):

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9}}$$

   Calculators can do this too! Here's what the Mathematica code looks like.

```
In[3]:=  Series[Log[1 + x], {x, 0, 9}]
```

$$\texttt{Out[3]=}\quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} + O[x]^{10}$$

Figure 4: Mathematica Code

   You should notice that the Mathematica command is of this form,

“Series[f, $\{x, \text{about } a, \text{degree } n\}$]”

   which will generates a power series expansion for f about the point a to order n. This “about a” business may seem strange, because all our examples have been about zero, but this will change, and we'll use different values for a.

2. Find the Taylor series for the function $f(x) = \ln(1+x)$.

   Work: Pattern is obvious.

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} + \cdots}$$

   Written as a an infinite sum[4] we have.

$$\boxed{\ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i}}$$

3. Use a graphing utility to graph the a sequence of Taylor polynomials and see if you can guess the interval of convergence.

   Work: Here's a graph of both $f(x) = \ln(1+x)$ (in black) and the ninth degree polynomial (in red). You should notice that the fit is not perfect, but it looks damn good for $-1 < x < 1$. Although not obvious, the higher degree for this polynomial the better the fit becomes, but only for $x \in (-1,\ 1)$.

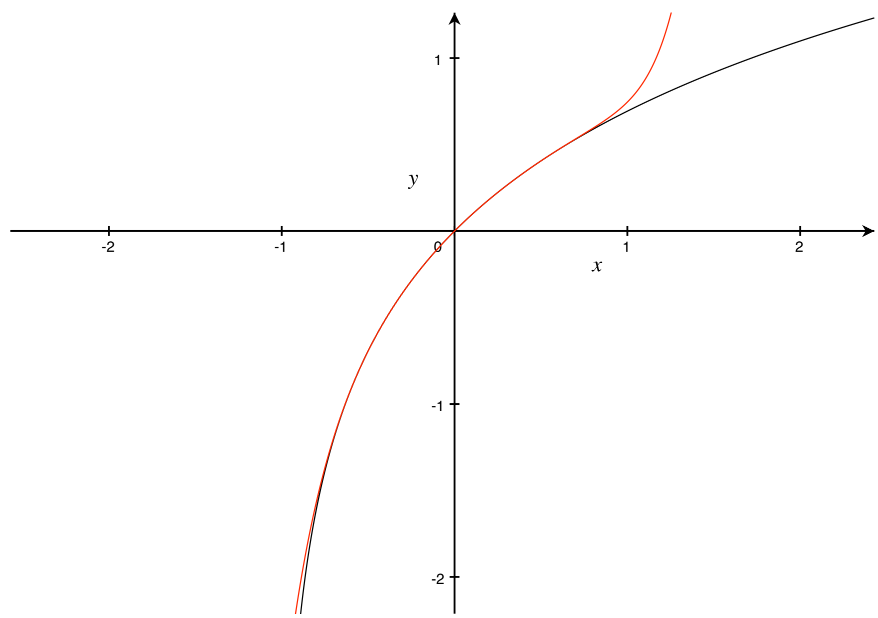

Figure 5: Looking good for $-1 < x < 1$.

   The interval of convergence is $\boxed{(-1,\ 1)}$

4. Find the Taylor polynomial about $x = 0$ for the function $f(x) = (1+x)^3$.

   Work: The binomial expansion gives

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3.$$

[4]You may want to review earlier material on using the summation symbol.

Now for the Taylor polynomial.

$$\begin{aligned}
f(x) &= (1+x)^3 \\
f'(x) &= 3\cdot(1+x)^2 \\
f''(x) &= 3\cdot 2\cdot(1+x) \\
f'''(x) &= 3\cdot 2\cdot 1 \\
f^{(4)}(x) &= 0 \\
\vdots &= \vdots \\
f^{(n)}(x) &= 0
\end{aligned}$$

Now, let's evaluate each of these derivatives at $x = 0$.

$$\begin{aligned}
f'(0) &= 3 \\
f''(0) &= 6 \\
f'''(0) &= 6 \\
f^{(4)}(0) &= 0 \\
\vdots &= \vdots \\
f^{(n)}(0) &= 0
\end{aligned}$$

So we have (after a bit of simplification):

$$\boxed{(1+x)^3 = 1 + 3x + 3x^2 + x^3}$$

This is certainly true for all x.

5. Use what you already know to *intuitively* show that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots.$$

Work: Looks like a simple substitution, where we have:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots$$

and we use $y = x^2$.

$$\begin{aligned}
e^y &= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots \\
e^{x^2} &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots
\end{aligned}$$

Q.E.D.

6. Use what you already know to *intuitively* show that

$$x^2\cos x^2 = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \cdots.$$

Work: Looks like a simple substitution, where we have:

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

and we use $y = x^2$.

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$
$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^12}{6!} + \cdots$$

Now just multiply both sides by x^2, and you'll get.

$$x^2 \cos x^2 = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \cdots .$$

Q.E.D.