MTH 122 — Calculus II

**Essex County College — Division of Mathematics and Physics**[1]

Lecture Notes #13 — Sakai Web Project Material

We won't finish this in class, but I do encourage you to continue working on this on your own. You should also try using a graphing utility to graph very high degree polynomial ... you might learn something!

Consider the functions

$$y = e^{-x^2} \qquad \text{and} \qquad y = \frac{1}{1+x^2}.$$

1. Write the Taylor expansions for the two functions about $x = 0$. What is similar about these two series? What is different?

2. Looking at the series, which function do you predict will be greater over the interval $(-1,\ 1)$? Graph both and see.

3. Are these functions *even* or *odd*? How might you see this by looking at the series expansion?

4. By looking at the coefficients, explain why it is reasonable that the series for

$$y = e^{-x^2}$$

converges for all values of $x$, but the series for

$$y = \frac{1}{1+x^2}$$

converges only on $(-1,\ 1)$.

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2ε. Last revised January 10, 2009.

1. Write the Taylor expansions for the two functions about $x = 0$. What is similar about these two series? What is different?

   **Answer:** The work is not shown, ask if you cannot do this.

$$
\begin{aligned}
e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots \\
\frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 - x^6 + x^8 - x^{10} + \cdots
\end{aligned}
$$

2. Looking at the series, which function do you predict will be greater over the interval $(-1,\ 1)$? Graph both and see.

   **Answer:** Well, just looking at the expansions above I'd have to say that the series for $y = \left(1 + x^2\right)^{-1}$ looks like it will be bigger. The reason is that the coefficients for each term in the series is 1, whereas the coefficients for $y = e^{-x^2}$ in the series expansion is getting *really small*, really quick! Here's a graph of of both, accurately scaled.

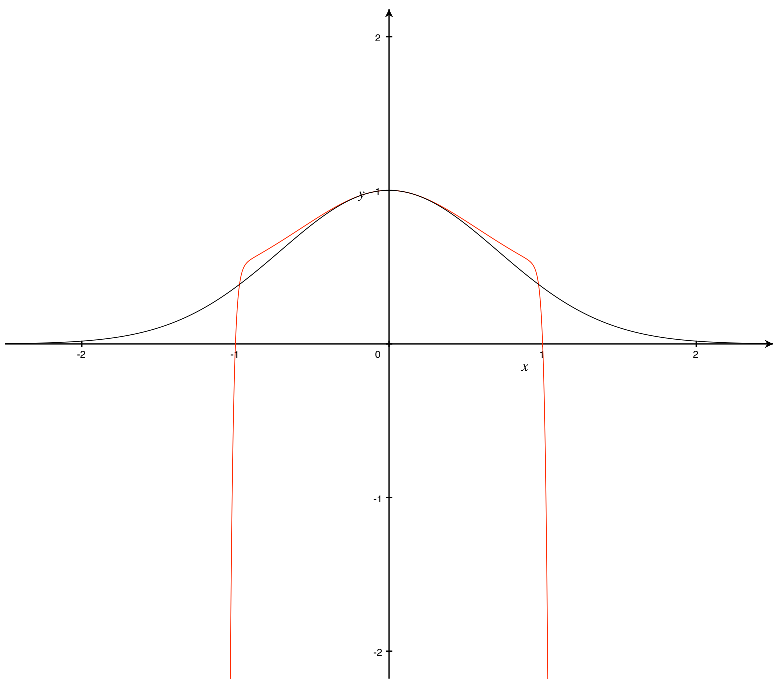


Figure 1: Label each graph! Yes, you can do it!

3. Are these functions *even* or *odd*? How might you see this by looking at the series expansion?

   **Answer:** Both graphs are even, no matter where we cut it. Recall the test for even symmetry is

$$f(x) = f(-x).$$

4. By looking at the coefficients, explain why it is reasonable that the series for

$$y = e^{-x^2}$$

converges for all values of $x$, but the series for

$$y = \frac{1}{1+x^2}$$

converges only on $(-1,\ 1)$.

**Answer:** Although not obvious, $y = e^{-x^2}$'s series converges for all $x$, whereas the series for $y = \left(1+x^2\right)^{-1}$ only converges for $x \in (-1,\ 1)$. Looking at the coefficients you should be able to convince your self that factorials grow faster than the powers.