MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #15 — Sakai Web Project Material

1 Don't Be Misled!

Last time we showed that

$$S = 10 + 100 + 1000 + \cdots$$

was negative, and we all agreed[2] that was clearly wrong, although our reasoning was quite sound. Basic to our understanding was that we are trying to extend our understanding of the finite to the infinite, without having a real grasp of how difficult this actually is. So, it is clearly easy to be mislead when dealing with the infinite, especially if we think our finite ways can be easily extended to the infinite. They can't!

The last example is quickly moving towards the infinite, and this negative result we derived is clearly wrong. Likewise, you should also note that we showed that $1 = 0.\bar{9}$ by looking at the infinite sum, it was geometric. Anyway, don't be misled into thinking that the sums of successively shrinking terms will always work out to be a finite number. In fact a very famous one is the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. This sum is actually unbounded and is said to diverge to infinity—although it is really slow moving. For Example

$$\begin{aligned}
\sum_{n=1}^{10^6} \frac{1}{n} &\approx 14.4 \\
\sum_{n=1}^{10^{12}} \frac{1}{n} &\approx 28.2 \\
\sum_{n=1}^{10^{24}} \frac{1}{n} &\approx 55.8 \\
\sum_{n=1}^{10^{100}} \frac{1}{n} &\approx 230.8 \\
\sum_{n=1}^{10^{1000}} \frac{1}{n} &\approx 2303.2
\end{aligned}$$

To see this we might make this simple observation (first made by a French Catholic Bishop named Oresme[3] in the fourteenth century) ...

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using $\LaTeX\,2_\varepsilon$. Last revised January 10, 2009.

[2] Well, I'd like to think so.

[3] Calvin C. Clawson, *Mathematical Mysteries* (Perseus Books, 1996), 61.

. . . where . . .

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

. . . is certainly greater than . . .

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$

. . . futhermore . . .

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots=1+\frac{1}{2}+\frac{1}{2}+\cdots$$

This may in fact be convincing, but remain leery of arguments like these. Being misled is always possible—and idiots and genius are equally susceptible to being duped. Truth in our own lives is often far more complicated, but it is surprising how 'black-and-white' most people are in their thinking. You may come to know a truth in your search for answers, but the truth may escape detection at first. Truth is usually hiding and it is almost never as obvious as it first appears. Do not become complacent and always distinguish between assumption and truth.

2 Introduction to Sequences and Series, Part II

1. **Definition**: A sequence $\{a_n\}$ is bounded above if there is a number M such that

   $$a_n \leq M,$$

   for all $n \geq 1$.

   A sequence $\{a_n\}$ is bounded below if there is a number m such that

   $$a_n \geq m,$$

   for all $n \geq 1$.

   If a sequence $\{a_n\}$ is bounded above and below, then it is a bounded sequence.

   Example The convergent sequence $\{n/(n+1)\}$ is bounded because $0 < a_n < 1$ for all n. The divergent sequence $\{(-1)^n\}$ is also bounded because $-1 \leq a_n \leq 1$ for all n.

2. **Definition**: Given a series

   $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

   let s_n denote its n^{th} partial sum:

   $$s_n = \sum_{i=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

   If the sequence $\{s_n\}$ is convergent and limit is s, then the series is convergent and its sum is s, otherwise we say the series divergent.

3. The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

If $|r| \geq 1$, the geometric series is divergent.

4. The p-series

$$\sum_{n=1}^{\infty} n^{-p}$$

is convergent if $p > 1$, and is divergent if $p \leq 1$.

5. If the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent, then

$$\lim_{n\to\infty} a_n = 0.$$

6. **Test for Divergence**: If $\lim_{n\to\infty} a_n \neq 0$, or $\lim_{n\to\infty} a_n = \text{DNE}$, then

$$\sum_{n=1}^{\infty} a_n$$

is divergent.

3 Examples

1. Fund the sum.

$$6 - 2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \cdots$$

2. Does the series

$$\sum_{n=1}^{\infty} \left(1 - e^{-n}\right)$$

converge? If so, what's the sum?

3. Does the series

$$\sum_{n=1}^{\infty} e^{-n}$$

converge? If so, what's the sum?

4. Show that

$$\int_{1}^{\infty} \frac{1}{x}\, \mathrm{d}x$$

diverges.

Now use the following graph to illustrate why the harmonic series diverges.

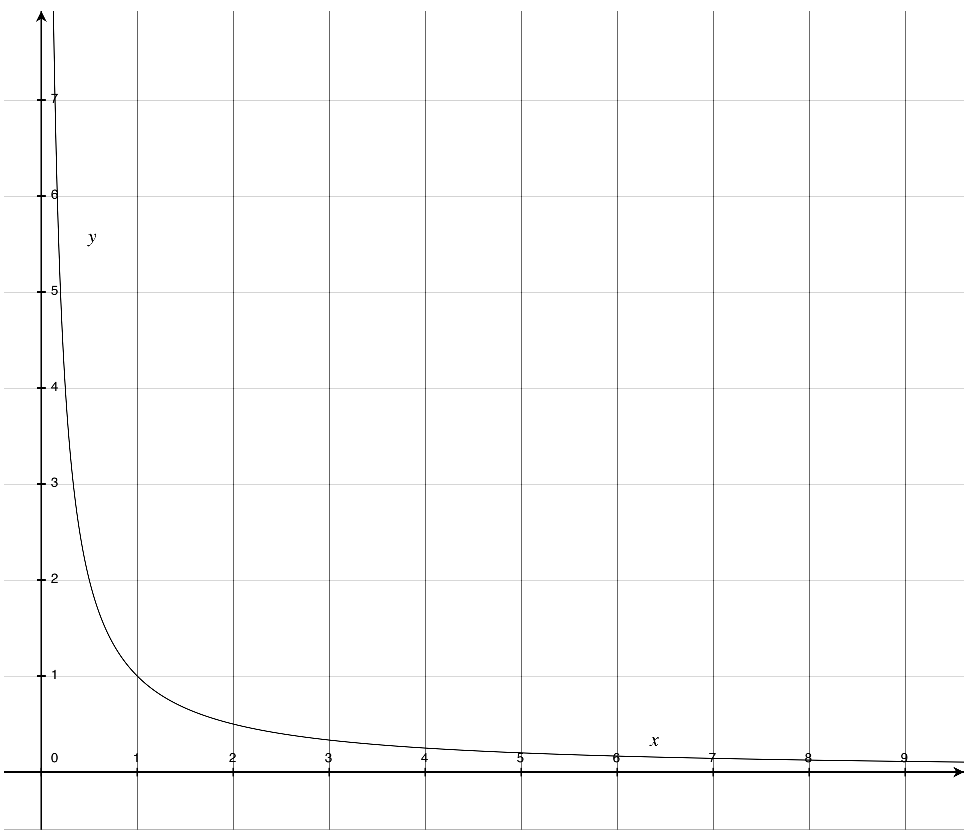

Figure 1: Partial graph of $y = 1/x$.