MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #16 — Sakai Web Project Material

1 Introduction to Sequences and Series, Part III

1. **Convergence Properties of Series**

   (a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, and if k is a constant, then

      i. $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

      ii. $\sum_{n=1}^{\infty} k a_n$ converges to $k \sum_{n=1}^{\infty} a_n$.

   (b) Changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does not converge.

   (c) If $\lim_{n\to\infty} a_n \neq 0$, or $\lim_{n\to\infty} a_n = \text{DNE}$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

   (d) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ diverges if $k \neq 0$.

2. **The Integral Test**: Suppose f is a continuous, positive, decreasing function on $[1,\ \infty)$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\ \mathrm{d}x$ is convergent. In other words:

   (a) If $\int_1^{\infty} f(x)\ \mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) If $\int_1^{\infty} f(x)\ \mathrm{d}x$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

   Example: For what values of p does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

   converge?

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2_ε. Last revised January 10, 2009.

Work: First just look at the limit

$$\lim_{n\to\infty} \frac{1}{n^p}$$

for $p \leq 0$. Now use the integral test on $0 < p \leq 1$ and $p > 1$.

1.1 Examples

1. Do the series converge or diverge?

   (a) [2]

   $$\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$$

   (b) [3]

   $$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

   (c) [4]

   $$\sum_{n=1}^{\infty} \frac{3}{n^2+4}$$

   (d) [5]

   $$\sum_{n=1}^{\infty} \frac{n+2^n}{n2^n}$$

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[2] Converges.

[3] Diverges.

[4] Converges.

[5] Diverges. **Hint:**

$$\int \frac{x+2^x}{x2^x}\ \mathrm{d}x = \ln x - \frac{1}{2^x \ln 2} + C$$

(e) [6]
$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+2}$$

2. Consider the series
$$\sum_{n=2}^{\infty} \ln \frac{(n-1)(n+1)}{n^2}.$$

(a) Show that S_4 is $\ln(5/8)$.

(b) Show that S_n is
$$\ln \frac{n+1}{2n}.$$

(c) Show that this series converges to
$$-\ln 2.$$

[6] Diverges.