MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #18 — Sakai Web Project Material

1 Introduction to Sequences and Series, Part V

1. The comparison test that we used prior, relies on verifying an inequality between a_n and b_n, however difficult this may be. To avoid this, we can instead use the following test.

   Limit Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$ is a finite positive number, then either both series converge or both series diverge.

   Use the limit comparison test to determine if the following series converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{n^2+6}{n^4-2n+3}$

   Work: Let $a_n = \frac{n^2+6}{n^4-2n+3}$, and since a_n behaves like $1/n^2$ (a convergent p-series) as $n \to \infty$, let $b_n = \frac{1}{n^2}$. We have

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4+6n^2}{n^4-2n+3} = 1.$$

   The limit comparison test applies with $c = 1$. Since the p-series converges, and $c = 1 > 0$, this test shows that

   $$\sum_{n=1}^{\infty} \frac{n^2+6}{n^4-2n+3}$$

   $\boxed{\text{converges}}$.

   (b) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

   Work: You may recall that $\sin x \approx x$ for x near zero. So as $n \to \infty$ the

   $$\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}.$$

   Let $a_n = \sin\left(\frac{1}{n}\right)$, and since a_n behaves like $1/n$ (a divergent p-series, called the harmonic series) as $n \to \infty$, let $b_n = \frac{1}{n}$. We have (this limit was done geometrically

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in calculus I)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

The limit comparison test applies with $c = 1$. Since the p-series diverges, and $c = 1 > 0$, this test shows that

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$\boxed{\text{diverges}}$.

2. **Alternating Series Test**: If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \qquad (a_n > 0)$$

satisfies

(a) $a_{n+1} \leq a_n$ for all n;

(b) and $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Example: Show that the alternating harmonic series is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Work: We need to show.

(a) $a_{n+1} \leq a_n$ for all n.

$$\frac{1}{n+1} < \frac{1}{n} \qquad \Rightarrow \qquad 0 < 1.$$

(b) and $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Since both conditions are satisfied, we can state that the alternating harmonic series $\boxed{\text{converges}}$. However, this test is for alternating series only.

3. **Definition**: The series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of the absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

4. **Definition**: The series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent, but not absolutely convergent.

5. **Theorem**: If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

   Determine if the series is absolutely convergent or conditionally convergent.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

   Work: This series is $\boxed{\text{absolutely convergent}}$ because the series converges and

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

   is a convergent p-series.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

   Work: This series is $\boxed{\text{conditionally convergent}}$ because the series converges and

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

   is a divergent p-series (harmonic series).

6. **The Ratio Test**:

   (a) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

   (b) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (c) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the ratio test fails, and we can draw no conclusion.

   Partial Proof I: Suppose

$$\sum_{n=1}^{\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1.$$

   Now let

$$0 < L < x < 1,$$

then for all sufficiently large n, we have

$$\frac{|a_{n+1}|}{|a_n|} = L < x < 1 \qquad \Rightarrow \qquad \frac{|a_{n+1}|}{|a_n|} < x.$$

Following forward, we have:

$$\begin{aligned} |a_{n+1}| &< |a_n| \cdot x \\ |a_{n+2}| < |a_{n+1}| \cdot x &< |a_n| \cdot x^2 \\ |a_{n+3}| < |a_{n+2}| \cdot x &< |a_n| \cdot x^3 \\ \vdots &< \vdots \\ |a_{n+i}| &< |a_n| \cdot x^i, \end{aligned}$$

which can be continued forever. This $|a_n|$ can be fixed at some value a, so we have

$$|a_{n+i}| < a \cdot x^i.$$

Since $0 < L < x < 1$ we know that the series

$$\sum_{i=1}^{\infty} a x^i$$

is a convergent geometric series, hence

$$\sum_{i=1}^{\infty} |a_{n+i}|$$

converges by comparison.

Partial Proof II: Suppose

$$\sum_{n=1}^{\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1.$$

That is, for some sufficiently large n, we have

$$|a_{n+1}| > |a_n|.$$

This, of course, leads to an increasing sequence, thus

$$\lim_{n \to \infty} a_n \neq 0,$$

so the series diverges.

Example: Show that

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges.

Work: Using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{1/\left(n+1\right)!}{1/n!}\\ &= \lim_{n\to\infty}\frac{n!}{(n+1)!}\\ &= \lim_{n\to\infty}\frac{1}{n+1}=0<1\end{aligned}$$

Since the limit is less than 1, we conclude by the ratio test that the series is convergent.

2 Examples

1. Show that[2]

$$\sum_{n=1}^{\infty}\frac{1}{n!}=e-1.$$

2. We know that the harmonic series diverges, but what does the ratio test tell us?

3. We also know that the alternating harmonic series converges, but what does the ratio test tell us?

[2]Use the series for e^x.

4. Use the limit comparison test[3] to see if

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

converges or diverges.

5. Use the limit comparison test[4] to see if

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

converges or diverges.

[3] Converges: Use a geometric series for comparison.
[4] Diverges: Use a p-series for comparison.

6. Show

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 3n}{4n-1}$$

diverges, by using the test for divergence. Also use the alternating series test to see what happens.

7. Given two convergent series

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} b_n$$

we know that the term-by-term sum

$$\sum_{n=1}^{\infty} (a_n + b_n)$$

converges. What about the series formed by taking the product of terms

$$\sum_{n=1}^{\infty} a_n \cdot b_n?$$

(a) Show that if $a_n = 1/n^2$ and $b_n = 1/n^3$, that

$$\sum_{n=1}^{\infty} a_n \cdot b_n$$

converges.

(b) Explain why

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges.

(c) Show that if if $a_n = b_n = (-1)^n / \sqrt{n}$, that

$$\sum_{n=1}^{\infty} a_n \cdot b_n$$

diverges.