MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #19 — Sakai Web Project Material

1 Introduction to Sequences and Series, Part VI

1. **The Root Test**:

   (a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

   (b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test fails, and we can draw no conclusion.

   This test works because

   $$\lim_{n\to\infty} \sqrt[n]{a_n} = r$$

   tells use that the series is comparable to a geometric series with ration r.

   Example: Use the root test to determine if the series converges.

   $$\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$$

   Work:

   $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

   So the series $\boxed{\text{converges}}$ absolutely.

   Example: Use the root test to determine if the series converges.

   $$\sum_{n=1}^{\infty} \left(\frac{5n^2+1}{3n^2}\right)^n$$

   Work:

   $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{5n^2+1}{3n^2}\right)^n} = \lim_{n\to\infty} \frac{5n^2+1}{3n^2} = \frac{5}{3} > 1$$

   So the series $\boxed{\text{diverges}}$.

[1] This document was prepared by Ron Bannon (ron.bannon@mathography.org) using LaTeX 2_ε. Last revised January 10, 2009.

2 Summary for Testing Series

Before you come to the exam, I strongly suggest that you review the following listed items. I also think it might be appropriate to make a one page review sheet covering these items, and this should be prepared well in advance of any exam/quiz related to this material. Representative problems are provided, but we'll only be doing a few of these in class.

1. Know the p-series!

   The p-series
   $$\sum_{n=1}^{\infty} n^{-p}$$
   is convergent if $p > 1$, and is divergent if $p \leq 1$.

2. Know the geometric series!

   The geometric series
   $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$
   is convergent if $|r| < 1$, and its sum is
   $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$
   If $|r| \geq 1$, the geometric series is divergent.

3. Know the test for divergence!

   If the series
   $$\sum_{n=1}^{\infty} a_n$$
   is convergent, then
   $$\lim_{n\to\infty} a_n = 0.$$

4. Know the integral test!

   The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\ \infty)$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\ \mathrm{d}x$ is convergent. In other words:

   (a) If $\int_1^{\infty} f(x)\ \mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) If $\int_1^{\infty} f(x)\ \mathrm{d}x$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

5. Know the (limit) comparison tests!

Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, and $0 < a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(b) If $\sum_{n=1}^{\infty} b_n$ is divergent, and $a_n \geq b_n > 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Limit Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$ is a finite positive number, then either both series converge or both series diverge.

6. Know the test for alternating series!

Alternating Series Test: If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \qquad (a_n > 0)$$

satisfies

(a) $a_{n+1} \leq a_n$ for all n;

(b) and $\lim_{n\to\infty} a_n = 0$,

then the series converges.

7. Know the ratio test!

The Ratio Test:

(a) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the ratio test fails, and we can draw no conclusion.

8. Know the root test!

The Root Test:

(a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test fails, and we can draw no conclusion.

Yes, these concepts have been extensively covered! And many examples were done in class, in addition to these examples the homework/textbook should help you understand their proper use. The more problems you do, the better you'll get.[2]

3 Examples

Although answers are provided, they solutions are not extensively worked out. Please, if you need to see complete work for any of these problems—*just ask!*

Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

   Work: $\boxed{\text{Converges}}$ by the **Root Test**. That is,

$$\lim_{n\to\infty} \sqrt[n]{\left(\sqrt[n]{2} - 1\right)^n} = 0 < 1.$$

2. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

   Work: This one is a bit tough.

$$\begin{aligned} (\ln n)^{\ln n} &= \left(e^{\ln(\ln n)}\right)^{\ln n} \\ &= \left(e^{\ln n}\right)^{\ln(\ln n)} \\ &= (n)^{\ln(\ln n)} \end{aligned}$$

   Now as $n \to \infty$ we can state that $\ln(\ln n) > 2$ for all $n > 1618$, so we have

$$\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}.$$

[2] I am reminded of a professor at Columbia University who never gave examples, yet he knew the theory well—his chair requested that he extemporaneously make-up an example, he was completely befuddle as a result. His exams were impossibly difficult and even his research assistant would inform us prior to the exams that the *professor* wouldn't be able to do these problems. Fact is, theory and proof don't necessarily make for great problem solvers. In short—*get to work*!

Since

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, we can conclude that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

also $\boxed{\text{converges}}$ by the **Comparison Test**.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}}{n^2}$

Work: We know that the harmonic sequence is decreasing for all $n \geq 1$, we can state that

$$\sqrt[n]{e} = e^{1/n} < e,$$

for all $n \geq 1$. And

$$\sum_{n=1}^{\infty} \frac{e}{n^2}$$

is a convergent p-series. Now, using the **Comparison Test** we can conclude that

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}}{n^2}$$

is $\boxed{\text{convergent}}$. You could have also use the **Integral Test**, which I think is probably easier to do.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Work: $\boxed{\text{Diverges}}$ by the **Limit Comparison Test**. That is,

$$a_n = \frac{1}{2n+1}, \ b_n = \frac{1}{n}, \ \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{2} > 0.$$

Since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, so does

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}.$$

The **Integral Test** would also be valid.

5. $\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{2}$

Work: $\boxed{\text{Diverges}}$ by the **Test for Divergence**. That is,

$$\lim_{n\to\infty} (-1)^n \sqrt[n]{2}$$

does not exist.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+5}$

Work: Using

$$f(x) = \frac{\sqrt{x}}{x+5}$$

which is a continuous, positive and decreasing for $x > 5$. So this series $\boxed{\text{converges}}$ by the **Alternating Series Test**. That is, for $n \geq 6$, $|a_n|$ is decreasing and

$$\lim_{n\to\infty} |a_n| = 0.$$

7. $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

Work: $\boxed{\text{Converges}}$ by the **Root Test**. That is,

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = 0 < 1.$$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$

Work: This is an alternating series where

$$|a_n| > |a_{n+1}|,$$

for $n \geq 2$, and

$$\lim_{n\to\infty} a_n = 0.$$

So this series $\boxed{\text{converges}}$ by the **Alternating Series Test**.

9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

Work: Let

$$f(x) = \frac{x^2}{e^{x^3}}.$$

Since f is a positive, continuous, decreasing function on $[1, \infty)$, we can apply the **Integral Test**. And

$$\int_2^\infty \frac{x^2}{e^{x^3}}\,\mathrm{d}x = \frac{1}{3e},$$

so we can conclude that the original series $\boxed{\text{converges}}$.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$

Work: This series $\boxed{\text{converges}}$ by the **Alternating Series Test**. That is, for $n \geq 8$, $|a_n|$ is decreasing and

$$\lim_{n\to\infty} |a_n| = 0.$$

11. $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$

Work: $\boxed{\text{Diverges}}$ by the **Limit Comparison Test**. That is,

$$a_n = \left(\sqrt[n]{2} - 1\right),\ b_n = \frac{1}{n},\ \lim_{n\to\infty} \frac{a_n}{b_n} = \ln 2 > 0.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent (harmonic series), so we can conclude that

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$$

also diverges.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

Work: Since $0 \leq n\cos^2 n \leq n$ for $n \geq 1$, we have

$$\frac{1}{n + n\cos^2 n} \geq \frac{1}{n+n} = \frac{1}{2n}.$$

Thus, since

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent (harmonic series), we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$$

$\boxed{\text{diverges}}$ by the **Comparison Test**.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

Work: Using the comparison

$$\left|\frac{\sin 2n}{1+2^n}\right| < \frac{1}{2^n},$$

we know that

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a convergent geometric series. So the original series $\boxed{\text{absolutely converges}}$ by the **Comparison Test**.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Work: Using the **Limit Comparison Test**, with

$$a_n = \frac{n^2+1}{n^3+1} \approx \frac{1}{n} = b_n$$

we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0,$$

and since $\sum_{n=1}^{\infty} b_n$ is a divergent (harmonic series) series, we conclude that our original series $\boxed{\text{diverges}}$.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 > 1,$$

so the series $\boxed{\text{diverges}}$. The **Test for Divergence** would also be valid.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

Work: Here we have

$$\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$$

Absolutely $\boxed{\text{converges}}$ by the **Root Test**. That is,

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4}{n}\right)^n} = 0 < 1.$$

17. $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{e} < 1,$$

so the series $\boxed{\text{converges}}$.

18. $\sum_{n=1}^{\infty} \sin n$

Work: $\boxed{\text{Diverges}}$ by the **Test for Divergence** since

$$\lim_{n\to\infty} \sin n$$

does not exist.

19. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$

Work: $\boxed{\text{Diverges}}$ by the **Test for Divergence**. That is,

$$\lim_{n\to\infty} \left| \frac{(-1)^n n}{n+2} \right| = 1.$$

20. $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

Work: $\boxed{\text{Converges}}$ by the **Comparison Test**. That is,

$$\frac{1}{n+3^n} < \left(\frac{1}{3}\right)^n,$$

for all $n \geq 1$. And

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

is a convergent **geometric series**.

21. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Work: Let

$$f(x) = \frac{1}{x\sqrt{\ln x}}.$$

Since f is a positive, continuous, decreasing function on $[2,\ \infty)$, we can apply the **Integral Test**. And

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}\ \mathrm{d}x = \infty,$$

so we can conclude that the original series $\boxed{\text{diverges}}$.

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

Work: This series $\boxed{\text{converges}}$ by the **Ratio Test**. That is,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{5} < 1.$$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2\cdot 5\cdot 8\cdot\ \cdots\ \cdot(3n+2)}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1,$$

so the series $\boxed{\text{converges}}$.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

Work: This series $\boxed{\text{converges}}$ by the **Alternating Series Test**. That is, for $n \geq 2$, $|a_n|$ is decreasing and

$$\lim_{n\to\infty} |a_n| = 0.$$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln n}{(n+1)^3}$

Work: First, we have

$$\frac{n\ln n}{(n+1)^3} < \frac{n\ln n}{n^3} = \frac{\ln n}{n^2}$$

for $n \geq 1$. Now let

$$f(x) = \frac{\ln x}{x^2}.$$

Since f is a positive, continuous, decreasing function on $[2,\ \infty)$, we can apply the **Integral Test**,

$$\int_2^\infty \frac{\ln x}{x^2}\ \mathrm{d}x = 1,$$

so we can conclude that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

converges. Now, using the **Comparison Test**, with

$$\frac{n \ln n}{(n+1)^3} < \frac{\ln n}{n^2}$$

for $n \geq 2$, since we know that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is a convergent using the **Integral Test**, so we can conclude that the original series also $\boxed{\text{converges}}$.

26. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n n^2}{n!}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1,$$

so the series $\boxed{\text{converges}}$.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{5^n}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{5} < 1,$$

so the series $\boxed{\text{converges}}$.

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Work: Using the comparison

$$\frac{\sqrt{n^2-1}}{n^3+2n^2+5} < \frac{1}{n^2},$$

for $n \geq 1$, and we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series. So the original series $\boxed{\text{converges}}$.

29. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

Work: This series $\boxed{\text{converges}}$ by the **Alternating Series Test**. That is, for $n \geq 2$, $|a_n|$ is decreasing and

$$\lim_{n\to\infty} |a_n| = 0.$$

30. $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Work: $\boxed{\text{Diverges}}$ by the **Test for Divergence**. That is,

$$\lim_{n\to\infty} n \sin \frac{1}{n} = 1 \neq 0.$$

31. $\displaystyle\sum_{n=1}^{\infty} \tan \frac{1}{n}$

Work: Using the **Limit Comparison Test**, with

$$a_n = \tan \frac{1}{n} < \frac{1}{n} = b_n$$

we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0,$$

and since $\sum_{n=1}^{\infty} b_n$ is a divergent (harmonic series) series, we conclude that our original series $\boxed{\text{diverges}}$.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

Work: $\boxed{\text{Converges}}$ by the **Limit Comparison Test**. That is,

$$a_n = \frac{\sin(1/n)}{\sqrt{n}}, \; b_n = \frac{1}{n^{3/2}}, \; \lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a convergent p-series, so we can conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$$

also converges.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1,$$

so the series $\boxed{\text{converges}}$.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

Work: Using the **Ratio Test**, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{5} < 1,$$

so the series $\boxed{\text{converges}}$.

35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$

Work: $\boxed{\text{Converges}}$ by the **Root Test**. That is,

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1} \right)^{n^2}} = \frac{1}{e} < 1.$$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

Work: You might need to look up the hyperbolic functions before proceeding. The necessary definition is:

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

So we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2}{e^n + e^{-n}}.$$

Although this rewriting is not necessary, I think it is *obvious* that we are now dealing with an alternating series, with

$$|a_n| > |a_{n+1}|,$$

and,

$$\lim_{n\to\infty} a_n = 0.$$

So the series $\boxed{\text{converges}}$ by the **Alternating Series Test**.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

Work: This series $\boxed{\text{diverges}}$ by the **Test for Divergence**. That is

$$\lim_{n\to\infty} \frac{5^n}{3^n + 4^n} = \lim_{n\to\infty} \frac{(5/4)^n}{(3/4)^n + 1} = \infty.$$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

Work: $\boxed{\text{Diverges}}$ by the **Root Test**. That is,

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \infty > 1.$$