MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #20 — Sakai Web Project Material

1 Power Series

1. A **power series** is a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   where x is a variable and the a_n's are constants called the coefficients of the series. The domain of this function is the set of all x for which this series is convergent.

2. A power series in $(x - b)$ is a power series centered at b, where b is a constant.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n = a_0 + a_1 (x-b) + a_2 (x-b)^2 + a_3 (x-b)^3 + \cdots$$

   where x is a variable and the a_n's are constants called the coefficients of the series. The domain of this function is the set of all x for which this series is convergent, and you should notice that this series always converges for $x = a$.

3. **Theorem**: For a given power series

$$\sum_{n=0}^{\infty} a_n (x-b)^n = a_0 + a_1 (x-b) + a_2 (x-b)^2 + a_3 (x-b)^3 + \cdots$$

   there are only three possibilities:

   (a) The series converges only when $x = b$. The **radius of convergence** is defined to be $r = 0$.

   (b) The series converges for all x. The **radius of convergence** is defined to be $r = \infty$.

   (c) There is a positive number R such that the series converges if $|x - b| < R$ and diverges for $|x - b| > R$. What happens at $|x - b| = R$ should also be examined. The **radius of convergence** is between $b - r$ and $b + r$, including any endpoints where the series converges.

4. **Theorem:** If the power series

$$\sum_{n=0}^{\infty} a_n (x-b)^n = a_0 + a_1 (x-b) + a_2 (x-b)^2 + a_3 (x-b)^3 + \cdots$$

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX 2ε. Last revised January 10, 2009.

has a radius of convergence $R > 0$, the the function defined by

$$f(x) = a_0 + a_1 (x-b) + a_2 (x-b)^2 + a_3 (x-b)^3 + \cdots = \sum_{n=0}^{\infty} a_n (x-b)^n$$

is differentiable (and therefore continuous) on the interval $(b-R,\ b+R)$.

5. **Method for Computing the Radius of Convergence:** To calculate the radius of convergence, r, for the power series

$$\sum_{n=0}^{\infty} a_n (x-b)^n ,$$

use the ratio test.

(a) If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

is infinite, then $r = 0$.

(b) If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0,$$

then $r = \infty$.

(c) If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = k\,|x-b| ,$$

where k is a constant, then $r = \frac{1}{k}$.

Now let's look at some key examples.

- The geometric series

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

is certainly recognizable, but let's do the **Ratio Test** to see where it converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1$$

So $-1 < x < 1$, but what if $x = \pm 1$? If $x = 1$ we get

$$f(1) = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots ,$$

which is clearly divergent. Now if $x = -1$ we get

$$f(-1) = \sum_{n=0}^{\infty} 1 = 1 - 1 + 1 - \cdots ,$$

which is also clearly divergent. So the **Radius of Convergence** is $R = 1$, and the **Interval of Convergence** is $(-1,\ 1)$.

- Here's another (unrecognizable) series

$$f(x) = \sum_{n=0}^{\infty} n!x^n = 1 + x + 2x^2 + 6x^3 + \cdots$$

and we will do the **Ratio Test** to see where it converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = (n+1)|x| = \infty,$$

however, if $x = 0$ this limit will be zero and the series will be convergent. Here we need to adopt a convention, that $(x-a)^0 = 1$ even if $x = a$, so

$$f(0) = 1.$$

So the **Radius of Convergence** is $R = 0$, and the **Interval of Convergence** is $[0,\ 0]$ or $\{0\}$ indicating a set with one element.

- Here's another (unrecognizable) series

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

and we will do the **Ratio Test** to see where it converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n(x-1)^{n+1}}{(n+1)(x-1)^n} \right| = \frac{n|x-1|}{n+1} = |x-1| < 1,$$

so the series will be convergent when $0 < x < 2$. We will still need to look at $x = 0$ and $x = 2$, because the **Ratio Test** is inconclusive here. When $x = 0$ we get

$$f(0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots,$$

which is an alternating harmonic series and hence convergent. Now we need to look at $x = 2$ where we get

$$f(2) = \sum_{n=1}^{\infty} \frac{(2-1)^n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

a divergent harmonic series. So the **Radius of Convergence** is $R = 1$, and the **Interval of Convergence** is $[0,\ 2)$.

- Here's another (unrecognizable) series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$$

and we will do the **Ratio Test** to see where it converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0$$

so the series will be convergent when $-\infty < x < \infty$. So the **Radius of Convergence** is $R = \infty$, and the **Interval of Convergence** is $(-\infty,\ \infty)$.

2 Can You Do These?

2.1 Problems

1. For what values of x is the series
$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$
convergent?

2. Show that the power series for
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
converges for all x.

3. The Bessel[2] function order zero is defined by the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}.$$
Find the interval of convergence, which is also the domain.

4. Determine the radius of convergence for
$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n}.$$

5. Determine the radius of convergence for
$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}.$$

6. Find the radius and interval of convergence of the series
$$1 + 2^2x^2 + 2^4x^4 + \cdots + 2^{2n}x^{2n} + \cdots$$

7. Find the radius and interval of convergence of the series
$$\sum_{n=0}^{\infty} \frac{n\,(x+2)^n}{3^{n+1}}.$$

[2] Daniel Bernoulli—the Swiss mathematician—defined defined and it was later generalized by the German Friedrich Bessel.

2.2 Solutions

1. For what values of x is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

convergent?

Work:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{3|x|\sqrt{n+1}}{\sqrt{n+2}} \\ &= 3|x| \end{aligned}$$

Using the **Ratio Test** we have $3|x|$ or $-1/3 < x < 1/3$. Since the **Ratio Test** is inconclusive at $x = \pm 1/3$, we need to see what happens at these values. Looking at $x = 1/3$ we have

$$f\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which is an alternating series and converges by the **Alternating Series Test**. For $x = -1/3$ we have

$$f\left(-\frac{1}{3}\right) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

which is divergent by the **Integral Test**. You could have also use the p-series for comparison to show that this is divergent. So the **Radius of Convergence** is $R = 1/3$, and the **Interval of Convergence** is $(-1/3,\ 1/3]$.

2. Show that the power series for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for all x.

Work:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 \end{aligned}$$

Q.E.D.

3. The Bessel[3] function order zero is defined by the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}.$$

Find the interval of convergence, which is also the domain.

Work:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2} [(n+1)!]^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1) x^2}{4 (n+1)^2} \right| \\ &= \frac{x^2}{4} \lim_{n\to\infty} \frac{1}{4 (n+1)^2} = 0 < 1 \end{aligned}$$

So the radius of convergence is infinite (all x), thus the domain of the Bessel function order zero is $\mathbb{R}$, that is the interval of convergence is $\mathbb{R}$.

4. Determine the radius of convergence for

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n}.$$

Work:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^n (x-1)^{n+1}}{n+1} \div \frac{(-1)^{n-1} (x-1)^n}{n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)^n (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} (x-1)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)(x-1) n}{n+1} \right| \\ &= |x-1| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x-1| < 1 \end{aligned}$$

This power series converges for $|x-1| < 1$ (*i.e.* $0 < x < 2$) and diverges for $|x-1| > 1$. If we check the endpoints we will see that the series converges for $0 < x \leq 2$. The radius of convergence is 1. You may recall that this is the series (centered at 1) for $\ln x$.

5. Determine the radius of convergence for

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}.$$

[3] Daniel Bernoulli—the Swiss mathematician—defined defined and it was later generalized by the German Friedrich Bessel.

Work:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!} \div \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n-1} x^{2n-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)\, x^2}{2n\,(2n+1)}\right| \\
&= x^2 \lim_{n\to\infty} \frac{1}{2n\,(2n+1)} = 0 < 1
\end{aligned}$$

This power series converges for all x. The radius of convergence is infinite. You may recall that this is the series for $\sin x$.

6. Find the radius and interval of convergence of the series

$$1 + 2^2x^2 + 2^4x^4 + \cdots + 2^{2n}x^{2n} + \cdots$$

Work:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{2(n+1)}x^{2(n+1)}}{2^{2n}x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|4x^2\right| \\
&= 4x^2 \lim_{n\to\infty} 1 = 4x^2 < 1
\end{aligned}$$

Hopefully you recall how to solve quadratic inequalities from MTH-119. That is, solving $4x^2 < 1$ and you'll get $-1/2 < x < 1/2$. At both endpoints you will get a divergent series, so the radius of convergence is $1/2$, an the interval of convergence is $-1/2 < x < 1/2$.

7. Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n\,(x+2)^n}{3^{n+1}}.$$

Work:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)\,(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,(x+2)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\,(x+2)}{3n}\right| \\
&= \frac{|x+2|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+2|}{3} < 1
\end{aligned}$$

Hopefully you recall how to solve absolute valued inequalities from MTH-119. That is, solving $|x+2| < 3$ and you'll get $-5 < x < 1$. At both endpoints you will get a divergent series, so the radius of convergence is 3, an the interval of convergence is $5 < x < 1$.