MTH 122 — Calculus II

**Essex County College — Division of Mathematics and Physics**[1]

Lecture Notes #21 — Sakai Web Project Material

# 1 Representations of Functions as Power Series

If you were to take the function

$$f(x) = \frac{1}{1-x}$$

and carry out long division (we'll do this is class) you'll see that it goes on forever and has a very nice pattern, namely

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n.$$

This, of course is the geometric series and it converges for $|x| < 1$. You should note that

$$f(x) = \frac{1}{1-x}$$

is defined for all $x \neq 1$, but for $-1 < x < 1$, $f(x)$ can be written as a power series,

$$\sum_{n=0}^{\infty} x^n.$$

Alhough $f(-2) = 1/3$ and is easy to compute, we could not use the power series to compute this, because when $x = -2$ the series is not convergent. And even if we wanted to know $f(1/2)$, which incidentally is 2, we wouldn't use the power series to do the computation. Rewriting functions as power series does have uses, but for now I am more concerned with finding a power series representation of a given function.

Okay to better understand what's going on here, let's return to a problem from introductory calculus. The tangent line approximation $L(x)$ is the best linear approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at $a$. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P_2(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure the approximation is a good one, we stipulate the following:

$$\begin{array}{rcll}
P_2(a) & = & f(a) & P_2 \text{ and } f \text{ should have the same value at } a. \\
P'(a) & = & f'(a) & P_2' \text{ and } f' \text{ should have the same value at } a. \\
P_2''(a) & = & f''(a) & P_2'' \text{ and } f'' \text{ should have the same value at } a.
\end{array}$$

This can, of course, go on *ad infinitum*. Let's take a concrete example.

[1] This document was prepared by Ron Bannon (`ron.bannon@mathography.org`) using LaTeX $2_\varepsilon$. Last revised January 10, 2009.

1. Find the quadratic approximation $P_2(x) = A + Bx + Cx^2$ to the function $f(x) = e^x$ that satisfies the above three conditions with $a = 0$. Graph $P_2$ and $f$ on the same axis. Does $P_2$ fit $f$ better than a tangent line in the region where $a = 0$?

   **Work and Solution:** We are given $f(x) = e^x$ and $P_2(x) = A + Bx + Cx^2$, and we need their derivatives.

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ P_2(x) &= A + Bx + Cx^2 \\ P_2'(x) &= B + 2Cx \\ P_2''(x) &= 2C \end{aligned}$$

   Now, using $a = 0$ we can determine the constants $A$, $B$ and $C$. In the functions first.

$$f(0) = e^0 = 1 = P_2(0) = A \qquad \Rightarrow \qquad A = 1$$

   Now in the first derivative.

$$f'(1) = e^0 = 1 = P_2'(0) = B \qquad \Rightarrow \qquad B = 1$$

   Now in the second derivative.

$$f''(0) = e^0 = 1 = P_2''(0) = 2C \qquad \Rightarrow \qquad C = \frac{1}{2}$$

   So, the function $P_2(x) = 1 + x + x^2/2$. Here's the graph.

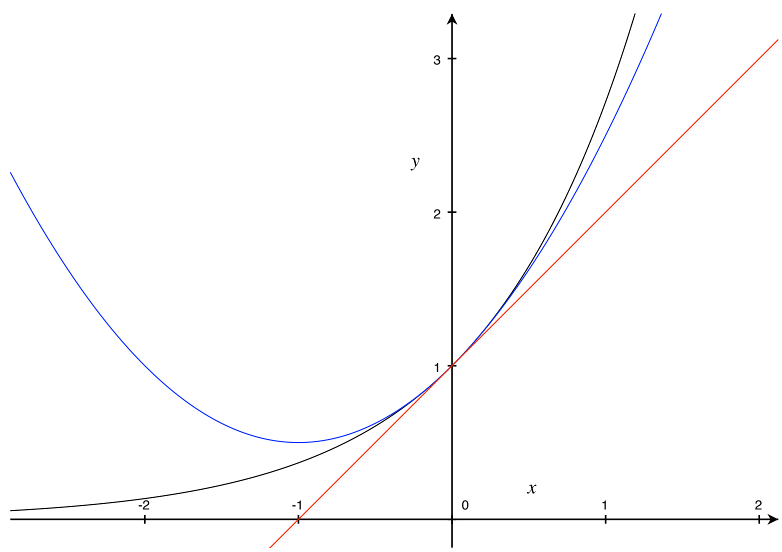


Figure 1: Partial graphs of the tangent line [red] to $e^x$ at $a = 0$, $P_2(x)$ [blue] and $f(x)$ [black].

   Yes, the quadratic is a better fit than is the linear function.

2. If we repeat this method for higher and higher-degree polynomials, we find that $f(x)$ can be better approximated. Now repeat this method until you get a forth degree polynomial, $P_4(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, that approximates $f(x)$ when $a = 0$. As before, graph $P_4$ and $f$ on the same axis. Does the forth degree $P_4$ fit $f$ better second degree $P_2$ in the region where $a = 0$?

**Work and Solution:** We are given $f(x) = e^x$ and $P_4(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, and we need their derivatives.

$$\begin{aligned}
f(x) &= e^x \\
f'(x) &= e^x \\
f''(x) &= e^x \\
f'''(x) &= e^x \\
f^{(4)}(x) &= e^x \\
P_4(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 \\
P_4'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 \\
P_4''(x) &= 2C + 6Dx + 12Ex^2 \\
P_4'''(x) &= 6D + 24Ex \\
P_4^{(4)}(x) &= 24E
\end{aligned}$$

Now, using $a = 0$ we can determine the constants $A$, $B$, $C$, $D$ and $E$. In the functions first.

$$f(0) = e^0 = 1 = P_4(0) = A \qquad \Rightarrow \qquad A = 1$$

Now in the first derivative.

$$f'(1) = e^0 = 1 = P_4'(0) = B \qquad \Rightarrow \qquad B = 1$$

Now in the second derivative.

$$f''(0) = e^0 = 1 = P_4''(0) = 2C \qquad \Rightarrow \qquad C = \frac{1}{2}$$

Now in the third derivative.

$$f'''(0) = e^0 = 1 = P_4'''(0) = 6D \qquad \Rightarrow \qquad D = \frac{1}{6}$$

Now in the forth derivative.

$$f^{(4)} = e^0 = 1 = P_4^{(4)}(0) = 24E \qquad \Rightarrow \qquad E = \frac{1}{24}$$

So, the function $P_4(x) = 1 + x + x^2/2 + x^3/6 + x^4/24$.[2] Here's the graph.

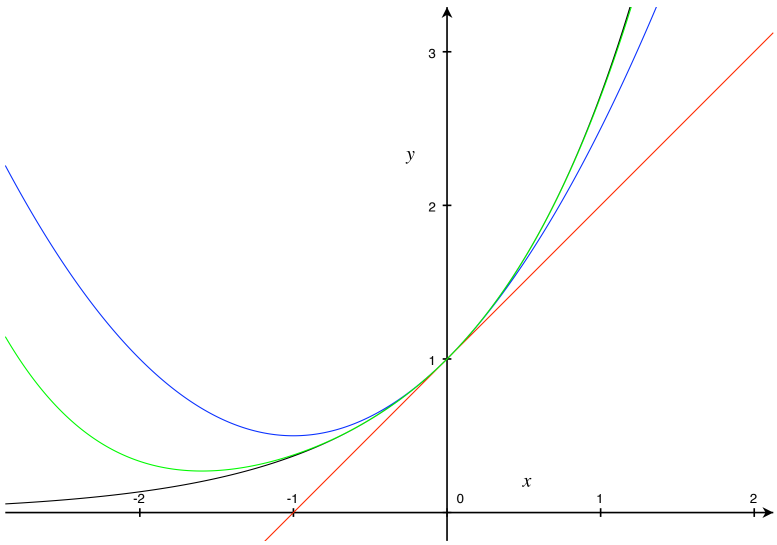


Figure 2: Partial graphs of $y = x + 1$ [red], $P_2(x)$ [blue], $P_4(x)$ [green] and $f(x)$ [black].

Yes, the forth degree polynomial is a better fit than is the second degree polynomial.

This, of course can be repeated *ad infinitum*. And the pattern for the $n^{\text{th}}$ degree polynomial is as follows:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!}.$$

And, in fact $e^x$ can be written as,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{i=0}^{\infty} \frac{x^n}{n!},$$

Which is a power series. You should also be able to verify that this power series is convergent for all $x$. So here we can state (unlike the opening example) that

$$e^x = \sum_{i=0}^{\infty} \frac{x^n}{n!}$$

is true for all $x$. That is, a transcendental function can be written as if it were a polynomial.

Okay, now let's do something a little dangerous here. We know that

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x,$$

so let's differentiate the power series term-by-term to see what we get.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \cdots\right] = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \cdots$$

Yes, that's incredible, but we nonetheless expected this.

[2] You should recognize the denominators. Yes, they're factorials.

3. Let's try to find the power series representation of

$$f(x) = \frac{1}{x+2}.$$

I'd like to suggest long division (we'll do this in class) to see if there's a pattern. Here's what we'll get

$$\frac{1}{x+2} = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n.$$

However, we could have also taking the approach that we used to find the power series for $e^x$, that is by taking derivatives. Furthermore, this example actually looks like our introductory problem,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n.$$

So let's rewrite this example to look like

$$\frac{1}{1-x}.$$

It doesn't even look possible at first site, but let's try!

$$\begin{aligned} \frac{1}{x+2} &= \frac{1}{2} \cdot \frac{1}{1+x/2} \\ &= \frac{1}{2} \cdot \frac{1}{1-(-x/2)} \end{aligned}$$

Now let's use what we know to see if we get the same result.

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n. \\ \frac{1}{2} \cdot \frac{1}{1-(-x/2)} &= 1 + (-x/2) + (-x/2)^2 + (-x/2)^3 + (-x/2)^4 + \cdots = \frac{1}{2} \cdot \sum_{n=0}^{\infty} (-x/2)^n . \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n. \end{aligned}$$

All as expected. Oh, the interval of convergence for this new power series is $(-2,\ 2)$. So unlike the power series for $e^x$ which is true for all $x$, equating the power series to this rational function

$$\frac{1}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

only holds for $(-2,\ 2)$.

## 2 Differentiation and Integration of Power Series

**Theorem** If the power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergnec $R > 0$, then the function $f$ defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable, and continuous on the interval $(a-R,\ a+R)$.

Basically we can differentiate the power series term-by-term as follows.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots\right] \\
f'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}[c_n(x-a)^n] = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\
&= \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}
\end{aligned}$$

Working backwards we can also integrate term-by-term too.

$$\begin{aligned}
\int f(x)\ \mathrm{d}x &= \int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]\ \mathrm{d}x = \int \left[c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots\right]\ \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \int [c_n(x-a)^n]\ \mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \\
&= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}
\end{aligned}$$

Let's take a simple example for integration. For this example I'd like to introduce two simple power series. The power series for sine is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

and the power series for cosine is:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Both of these series converge for all $x$. We know that

$$\int \cos x\ \mathrm{d}x = \sin x + C,$$

so let's give it a try using the series instead.

$$\begin{aligned} \int \cos x \ \mathrm{d}x &= \sin x + C \\ \int \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] \mathrm{d}x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + C \end{aligned}$$

Yes, exactly as expected. Now if were differentiate sine we should get cosine.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[\sin x] &= \cos x \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \end{aligned}$$

Yes, as expected once again.

Here's another example, but this one is related to

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n.$$

If we differentiate this series we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + x + x^2 + x^3 + x^4 + \cdots\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} x^n\right] \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, x^n \end{aligned}$$

Now we have a series expansion for

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, x^n.$$

The interval of convergence is $(-1,\ 1)$. It might be useful to graph

$$f(x) = \frac{1}{(1-x)^2},$$

and finite number of terms from the power series, let's say

$$p(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^9 + 9x^{10} + 10x^{11},$$

on the same axis to see how they fit each other. We're expecting a good fit only on the interval $(-1,\ 1)$. Certainly, taking more terms would create a better fit, but the point here is that we're getting a fit only on the interval $(-1,\ 1)$. You might want to play around with graphing $f(x)$ and various $n$'s for

$$\sum_{i=0}^{n} (i+1)\, x^i$$

to see what happens as $n$ increases.

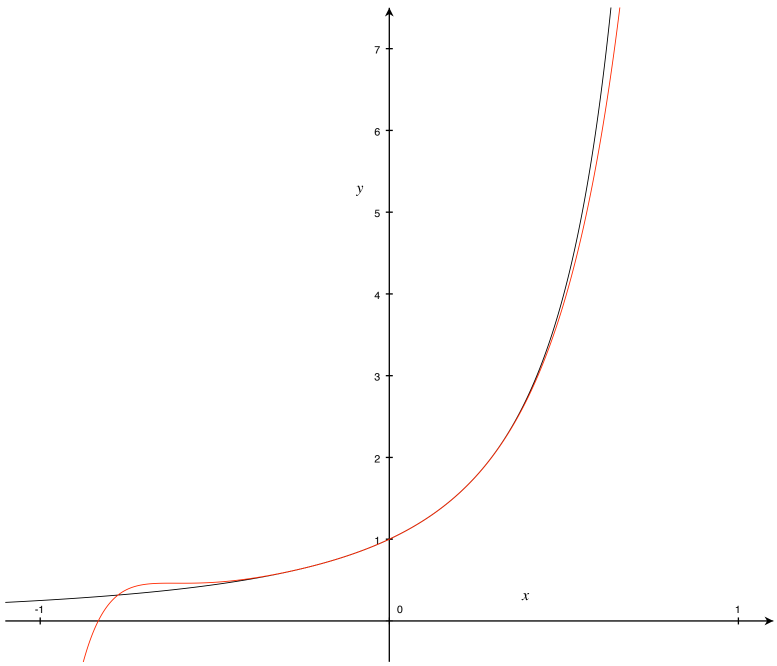


Figure 3: Partial graphs of $f(x)$ [black], $p(x)$ [red]. Not drawn to proper aspect ratio.

Tricks abound, and finding power series can be quite elegant, or thuggish. Well, for example, knowing the power series for

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

can lead us to the power series for

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots$$

And since we know (possibly remember) that

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C,$$

we can find a power series for arctangent by integrating the power series.

$$\begin{aligned}
\int \frac{1}{1+x^2}\,\mathrm{d}x &= \arctan x + C \\
\int \left(1 - x^2 + x^4 - x^6 + \cdots\right)\,\mathrm{d}x &= \arctan x + C_1 \\
C_2 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots &= \arctan x + C_1 \\
C_3 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots &= \arctan x
\end{aligned}$$

To find the constant let's take an easy value for $x$, let's say $x = 0$.

$$\begin{aligned} C_3 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots &= \arctan x \\ C_3 + 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \cdots &= \arctan 0 \\ C_3 &= 0 \end{aligned}$$

So now we can say that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which converges for $[-1,\ 1]$. Here's the graph of arctangent and the first ten terms (degree 19) of its power series.

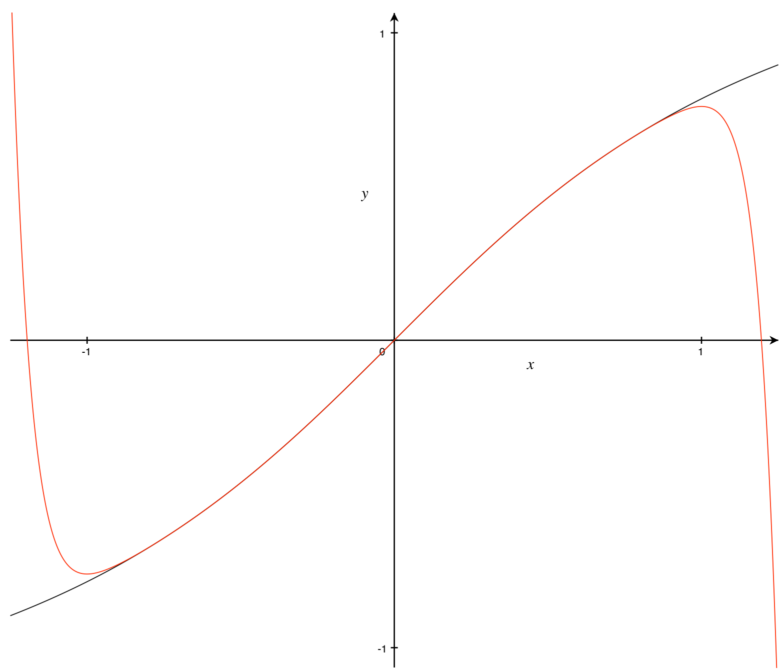


Figure 4: Partial graphs of $\arctan x$ [black], and the first ten terms of its power series [red].

Here you might want to note[3] that

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

[3]Leibniz's famous formula for finding $\pi$. You may want to compare this one with Srinivasa Ramanujan's formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26390n)}{(n!)^4\, 396^{4n}}.$$

Although Leibniz's famous formula looks easier, it's converges much slower than Srinivasa Ramanujan's formula. You might want to try computing three terms of each series to see the difference. If you're going to go *mad* and compute $\pi$ as an obsession, I strongly suggest you use Srinivasa Ramanujan's formula. The Japanese use a variation of Ramanujan's power series, and they, among others, are obsessed with finding digits of $\pi$. I believe the Japanese are well beyond 1,241,100,000,000 decimal digits. That's scary.

## 3 Examples

1. Use long division to find the power series for

$$f(x) = \frac{1}{1-x^4},$$

   and indicate the radius of convergence.

2. Use what you already know about the power series for

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

   to verify the power series obtain above.

3. Find the power series for

$$\frac{1}{1+x^7}$$

   and then use this power series to find a power series for

$$\int \frac{1}{1+x^7}\, \mathrm{d}x.$$

   Also, state the interval of convergence for each series.

4. We know that

$$\tan x = \frac{\sin x}{\cos x},$$

   and we know that

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \end{aligned}$$

   Try, using long divsion, to find the power series for tangent.

5. Use the method of fitting higher degree polynomials (see problem on $e^x$) to find the power series for $f(x) = \ln(1-x)$.

## 4 Answers

1. Use long division to find the power series for

$$f(x) = \frac{1}{1-x^4},$$

   and indicate the radius of convergence.

   **Answer:** The division is quite simple, but we'll do this in class.

$$f(x) = \frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + \cdots = \sum_{n=0}^{\infty} x^{4n}$$

   And the interval of convergence is $(-1,\ 1)$.

2. Use what you already know about the power series for

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

to verify the power series obtain above.

**Answer:** First, just rewrite

$$\frac{1}{1-x^4} = \frac{1}{1-(x^4)},$$

and then just plug it in!

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{1-(x^4)} &= 1 + x^4 + x^8 + x^{12} + \cdots
\end{aligned}$$

3. Find the power series for

$$\frac{1}{1+x^7}$$

and then use this power series to find a power series for

$$\int \frac{1}{1+x^7}\, \mathrm{d}x.$$

Also, state the interval of convergence for each series.

**Answer:** First, just rewrite

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)},$$

and then just plug it in!

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{1-(-x^7)} &= 1 - x^7 + x^{14} - x^{21} + \cdots = \sum_{n=0}^{\infty} (-1)^n\, x^{7n}
\end{aligned}$$

This series has an interval of convergence $(-1,\ 1)$. Using this power series to integrate:

$$\begin{aligned}
\int \frac{1}{1+x^7}\, \mathrm{d}x &= \int \left(1 - x^7 + x^{14} - x^{21} + \cdots\right)\, \mathrm{d}x \\
&= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \\
&= C + \sum_{n=0}^{\infty} (-1)^n\, \frac{x^{7n+1}}{7n+1}
\end{aligned}$$

and this series has an interval of convergence $(-1,\ 1)$

4. We know that

$$\tan x = \frac{\sin x}{\cos x},$$

and we know that

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \end{aligned}$$

Try, using long divsion, to find the power series for tangent.

**Answer:** This is torture! But you might be able to see a pattern if you start the division.

$$\frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

Yikes, if there's a pattern, I don't see it! Anyway, you can always look them up.

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n} \left(-4\right)^n \left(1 - 4^n\right)}{(2n)!} x^{2n-1}$$

The interval of convergence for this series is $(-\pi/2,\ \pi/2)$. The $B_{2n}$ are Bernoulli numbers. The main point I want to make here is that power series in general can be a pain to generate, but if you know how to use a computer you be able to it quickly!

5. Use the method of fitting higher degree polynomials (see problem on $e^x$) to find the power series for $f(x) = \ln(1-x)$.

**Answer:** We are given $f(x) = \ln(1-x)$ and suppose we're looking for a polynomial of degree six that fits this, so let$P_6(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6$, and the derivatives of both $f$ and $P_6$

$$\begin{aligned} f(x) &= \ln(1-x) \\ f'(x) &= -(1-x)^{-1} \\ f''(x) &= -(1-x)^{-2} \\ f'''(x) &= -2(1-x)^{-3} \\ f^{(4)}(x) &= -6(1-x)^{-4} \\ f^{(5)}(x) &= -24(1-x)^{-5} \\ f^{(6)}(x) &= -120(1-x)^{-6} \\ P_6(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 \\ P_6'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + 6Gx^5 \\ P_6''(x) &= 2C + 6Dx + 12Ex^2 + 20Fx^3 + 30Gx^4 \\ P_6'''(x) &= 6D + 24Ex + 60Fx^2 + 120Gx^3 \\ P_6^{(4)}(x) &= 24E + 120Fx + 360Gx^2 \\ P_6^{(5)}(x) &= 120F + 720Gx \\ P_6^{(6)}(x) &= 720G \end{aligned}$$

Now, using $x = 0$ we can determine the constants $A$, $B$, $C$, $D$, $E$, $F$, and $G$.

$$\begin{aligned}
P_6(0) &= f(0) = 0 = A \\
P_6'(0) &= f'(0) = -1 = B \\
P_6''(0) &= f''(0) = -1 = 2C \\
P_6'''(0) &= f'''(0) = -2 = 6D \\
P_6^{(4)}(0) &= f^{(4)}(0) = -6 = 24E \\
P_6^{(5)}(0) &= f^{(5)}(0) = -24 = 120F \\
P_6^{(6)}(0) &= f^{(6)}(0) = -120 = 720G
\end{aligned}$$

We're getting a fairly nice pattern, and I'm going to make a *guess*[4] here and say that

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

The interval of convergence is $[-1,\ 1)$. Let's look at the graph of $f(x)$ and $-\sum_{n=1}^{20} \frac{x^n}{n}$.

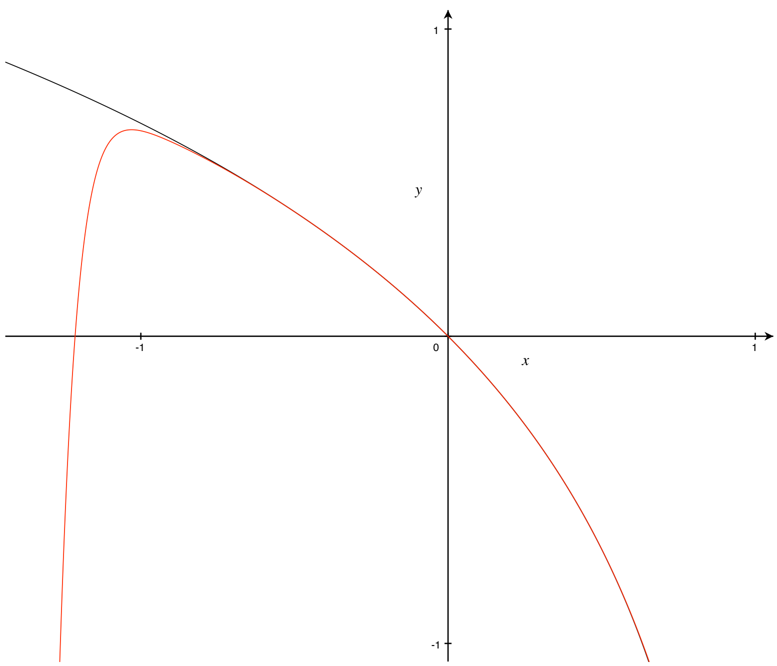


Figure 5: Partial graphs of $f(x)$ [black], and the first twenty terms of its power series [red].

Here we have (seen before) that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots$$

[4] We'll discuss the actual method soon.