MTH 122 — Calculus II

Essex County College — Division of Mathematics and Physics[1]

Lecture Notes #22 — Sakai Web Project Material

1 Finding Power Series

We certainly know by now that some functions can be written as power series. For example, we used the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1$$

to generate[2] many related power series. We also used curve fitting and derivatives to find other power series. For example, we now know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Not that we need to remember these, or even generate them quickly, but we do need to realize that we were able to find power series for some functions. However, we also had painfully difficult examples[3] that didn't work out so nicely.

So let's suppose we have a function $f(x)$ that has a power series expansion that is centered at $x = c$ and is valid for all x in an interval. It will be of this form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + +a_3(x-c)^3 + \cdots .$$

And we can differentiate this series term-by-term to get

$$\begin{aligned}
f(x) &= = a_0 + a_1(x-c) + a_2(x-c)^2 + +a_3(x-c)^3 + \cdots \\
f'(x) &= = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots \\
f''(x) &= = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + 20a_5(x-c)^3 + \cdots \\
\vdots &= \vdots \\
f^{(n)}(x) &= = n!a_n + \text{ all terms that follow will have a factor of } (x-c).
\end{aligned}$$

[1] This document was prepared by Ron Bannon (ron.bannon@mathography.org) using LaTeX 2ε. Last revised January 10, 2009.

[2] Examples involved integration, differention, and substitution.

[3] The tangent for example.

Now if we evaluate these functions at $x = c$ we'll get:

$$\begin{aligned} f(c) &= = a_0 \\ f'(c) &= = a_1 \\ f''(c) &= = 2a_2 \\ \vdots &= \vdots \\ f^{(n)}(c) &= = n!a_n \end{aligned}$$

Here we have a fairly simple way to generate the coefficients, that is, as long as the c is easy to evaluate in f and the derivatives of f are easy to find.

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Let's take a really simple example, let $f(x) = e^x$ and $c = 0$.

Work: Here we have

$$f(x) = e^x = f^{(n)}(x),$$

and if $c = 0$ we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}.$$

So our power series for e^x centered at zero is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The interval of convergence here is all reals.

The reason this was so easy to do is that e^x is easy to differentiate and evaluate at zero. But what about finding the power series for $f(x) = \sin x$ and $c = 0$.

Work: Again, let's start with the derivatives

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

By just taking four derivatives I hope you can see we're in an infinite loop. Let's evaluate now to what happens.

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \end{aligned}$$

So the coefficients are following a patern

$$0,\ 1,\ 0,\ -1,\ 0,\ \cdots$$

So our power series for $\sin x$ centered at zero is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

The interval of convergence here is all reals.

Okay, that was a little more difficult, but just to make sure you've got it, let's try finding the power series for $f(x) = \sin x$ and $c = 0$.

Work: Again, let's start with the derivatives

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

By just taking four derivatives I hope you can see we're in an infinite loop. Let's evaluate now to what happens.

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \end{aligned}$$

So the coefficients are following a patern

$$1,\ 0,\ -1,\ 0,\ 1,\ \cdots$$

So our power series for $\cos x$ centered at zero is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The interval of convergence here is all reals.

The general form is

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

and the power series is referred to as a **Taylor series**, and if $c = 0$ the Taylor series is referred to as a **Maclaurin series**.

A good way to find other series is to start with a known series and use the following operations: multiplication, substitution, differentiation, or integration to generate others. For example, given that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

do the following.

1. Find the series for $x^2 e^x$.

   Work: A very simple multiplication.

$$\begin{aligned} x^2 e^x &= x^2 \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) \\ &= x^2 + x^3 + \frac{x^4}{2!} + \cdots + \frac{x^{n+2}}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \end{aligned}$$

2. Find the series for $e^x \sin x$.

   Work: Well, this is not a simple multiplication, but let's try anyway.

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} + \cdots \end{aligned}$$

   Not easy[4] but you get the idea. Anyway, I don't see a simple pattern, but sometimes we're just looking for the a finite number of terms.

3. Find the series for $\cos\left(x^2\right)$

   Work: Very easy, just make a substitution.

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \cos\left(x^2\right) &= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \end{aligned}$$

[4] I also used a calculator to do this!

4. We know that[5]

$$\ln(1+x) = \int \frac{1}{1+x}\, \mathrm{d}x,$$

and that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Use this information to find the power series for $\ln(1+x)$.

Work:

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x}\, \mathrm{d}x \\ &= \int 1 - x + x^2 - x^3 + \cdots \, \mathrm{d}x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, x^n}{n} \end{aligned}$$

2 Some Maclaurin Series and Their Radii of Convergence

I'm not saying that you should memorize this list, and even if you do you may then go on to forget it. But, these series are all easily derivable and you should be able to derive each one, except the binomial.[6] However, please keep in mind that knowing a little can go a far way towards knowing a lot!

(a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ R = 1$

(b) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!},\ R = \infty$

(c) $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n\, x^{2n+1}}{(2n+1)!},\ R = \infty$

(d) $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n\, x^{2n}}{(2n)!},\ R = \infty$

(e) $\arctan x = \sum_{n=0}^{\infty} \dfrac{(-1)^n\, x^{2n+1}}{(2n+1)},\ R = 1$

(f) $(1+x)^k = \sum_{n=0}^{\infty} {}_kC_n\, x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3,\ R = 1$

[5] I omitting the constant here. That is I am saying that the constant is zero.

[6] The Binomial that you learned about in pre-calculus only works for natural numbers. This one here is referred to as Newton's generalization.

3 Examples

1. Find the power series for
$$f(x) = \frac{\sin x}{x},$$
and graph both. Be sure to use at least five terms of the power series.

2. Use long division to find the first three terms of the power series for
$$f(x) = \frac{x}{\sin x},$$
and graph both.

3. Newton was known for using power series to integrate. So in honor of Newton I'd like for use a power series to integrate
$$\int e^{x^2} \, dx.$$

4. Although difficult, we found that
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots .$$
Use this to find the following limit.
$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

5. Find the Maclaurin series for
$$f(x) = e^{-x^2} + \cos x,$$
and graph both f and good number (at least degree six) of initial terms of its Maclaurin series.

6. Find the Taylor series for f centered at 4 if
$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n (n+1)}$$
and its radius of convergence.

7. Use the binomial series
$$(1+x)^k = \sum_{n=0}^{\infty} {}_kC_n \, x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3, \; R = 1$$
to expand
$$(1-x)^{2/3}.$$
Graph both $y = (1-x)^{2/3}$ and at least the first ten terms of its power series.

4 Answers

1. Find the power series for

$$f(x) = \frac{\sin x}{x},$$

and graph both. Be sure to use at least five terms of the power series.

Work:

$$\begin{aligned} \frac{\sin x}{x} &= \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \end{aligned}$$

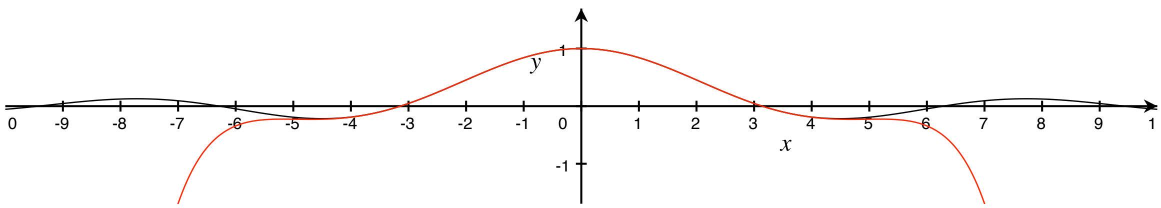

Figure 1: Partial graphs of $f(x)$ [black], and the first five terms of its power series [red].

2. Use long division to find the first three terms of the power series for

$$f(x) = \frac{x}{\sin x},$$

and graph both.

Work: Using long division I found.

$$f(x) = \frac{x}{\sin x} \approx 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$$

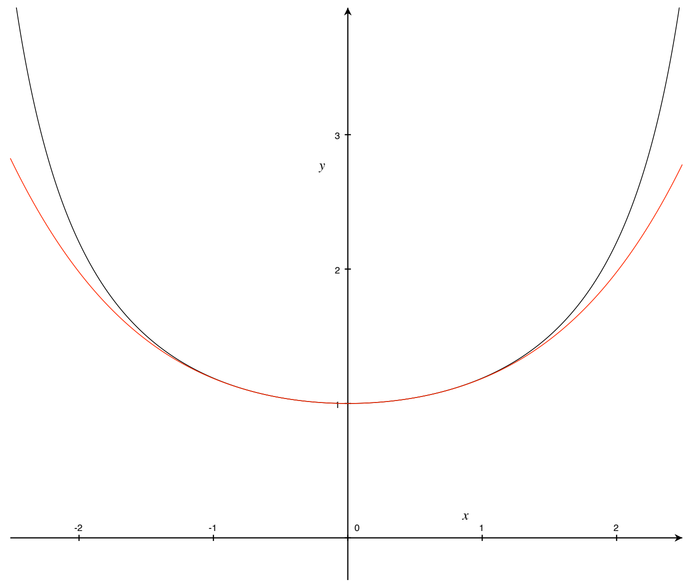

Figure 2: Partial graphs of $f(x)$ [black], and the first three terms of its power series [red].

3. Newton was known for using power series to integrate. So in honor of Newton I'd like for use a power series to integrate

$$\int e^{x^2}\, \mathrm{d}x.$$

Work:

$$\begin{aligned}\int e^{x^2}\, \mathrm{d}x &= \int 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \, \mathrm{d}x \\ &= C + x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!}\end{aligned}$$

4. Although difficult, we found that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots .$$

Use this to find the following limit.

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

Work: You could also use l'Hospital's Rule three times, but I want you to try using series first!

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots - x}{x^3}\\ &= \lim_{x\to 0}\frac{\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots}{x^3}\\ &= \lim_{x\to 0}\left(\frac{1}{3}+\frac{2x^2}{15}+\frac{17x^4}{315}+\frac{62x^6}{2835}+\cdots\right)=\frac{1}{3}\end{aligned}$$

Just for fun you should try using l'Hospital's Rule three times to if it leads to the same result.[7]

5. Find the Maclaurin series for

$$f(x) = e^{-x^2}+\cos x,$$

and graph both f and good number (at least degree six) of initial terms of its Maclaurin series.

Work:

$$\begin{aligned}\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\\ e^{-x^2} &= 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}-\cdots\\ e^{-x^2}+\cos x &= 2-\frac{3x^2}{2!}+\frac{13x^4}{4!}-\frac{121x^6}{6!}+\cdots+(-1)^n\left(\frac{1}{n!}+\frac{1}{(2n)!}\right)x^{2n}\end{aligned}$$

The graph that follows with $f(x)=e^{-x^2}+\cos x$ in black and

$$\sum_{n=0}^{100}(-1)^n\left(\frac{1}{n!}+\frac{1}{(2n)!}\right)x^{2n}$$

in red. I did not type all 100 terms, but instead simply entered the sum as indicated ($n = 100$). Whatever computer you use you should have some tools available that can help graph such expressions. Instead of being told what to use, you should try to figure out what works best for you.

6. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4)=\frac{(-1)^n\, n!}{3^n(n+1)}$$

and its radius of convergence.

[7] It does!

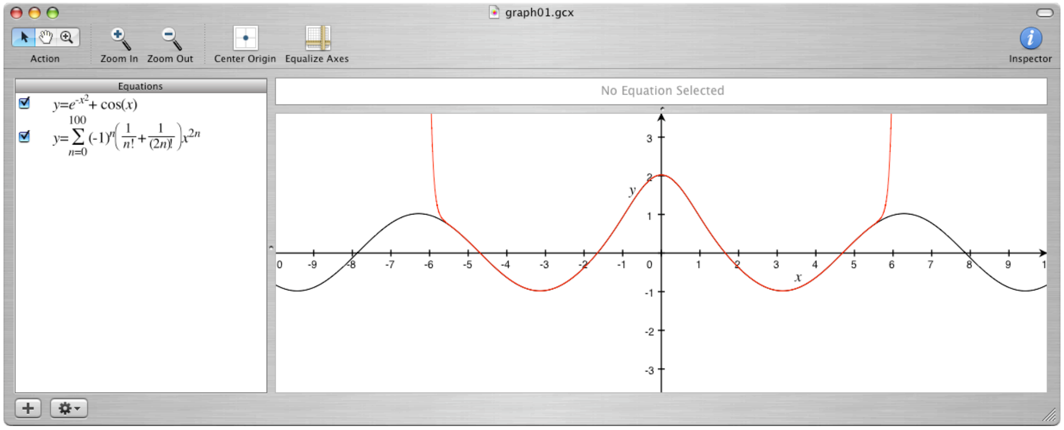

Figure 3: Here is what Grapher looks like [Mac OS X].

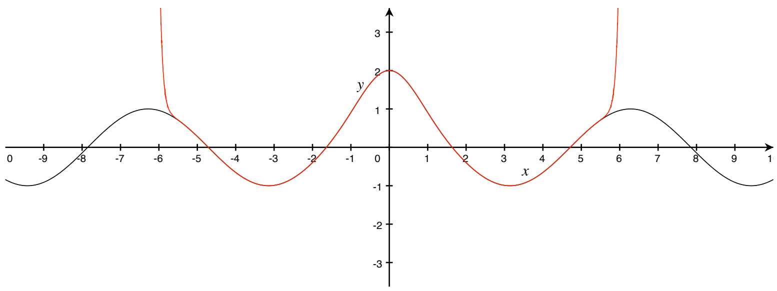

Figure 4: Partial graphs of $f(x)$ [black], and the first 100 terms of its power series [red].

Work: The general form of this Taylor series is

$$\begin{aligned} f(x) &= f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2!} + \frac{f'''(4)(x-4)^3}{3!} + \cdots + \frac{f^{(n)}(4)(x-4)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n (n+1) n!} (x-4)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n (n+1)} (x-4)^n \end{aligned}$$

Now, using the **Ratio Test**, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n (n+1)}{(-1)^n (x-4)^n} \right| \\ &= \lim_{n\to\infty} |x-4| \frac{(n+1)}{3(n+2)} \\ &= \frac{1}{3}|x-4| < 1 \end{aligned}$$

So the radius of convergence is 3.

7. Use the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} {}_kC_n \; x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3, \;\; R = 1$$

to expand

$$(1-x)^{2/3}.$$

Graph both $y = (1-x)^{2/3}$ and at least the first ten terms of its power series.

Work:

$$\begin{aligned}
(1+(-x))^{2/3} &= 1 + \frac{2}{3}(-x) + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}(-x)^2 + \frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}(-x)^3 + \cdots \\
&= 1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3 - \frac{7}{243}x^4 - \frac{14}{729}x^5 \cdots \\
&= 1 - \frac{2}{3}x - 2\sum_{n=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-5)}{3^n n!}x^n
\end{aligned}$$

Don't worry about the last line of this power series, they're often a real *pain* to find these forms, and many time were just concerned a finite number of terms.

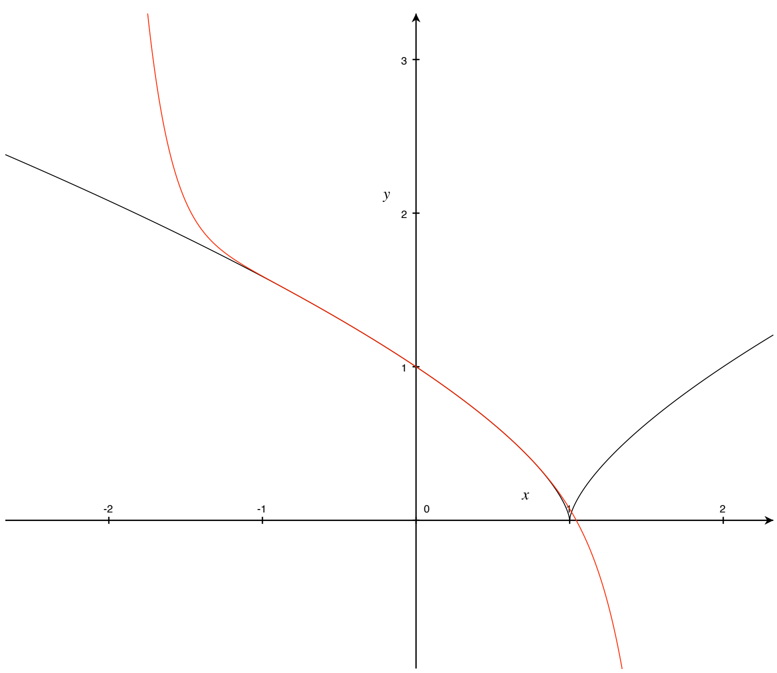

Figure 5: Partial graphs of $f(x) = (1-x)^{2/3}$ [black], and the first 12 terms of its power series [red].