**Name:** ______________________________

**Signature:** ______________________________

Show all work clearly and in order, and box your final answers. Justify your answers whenever possible. You have 20 minutes to take this 10 point quiz.

The number $\pi$ is often approximated, and many students believe that $\pi$ can be written down in some numerical form, usually as a decimal. Some say it's about 3 while others might go on to say 3.14, or possibly something more precise. Of special note is Akira Haraguchi, a mental health counsellor from Japan, that can recite than 100,000 decimal digits of $\pi$ from memory. However, no matter how precise you go, they're all just approximations.

One can find many exact expressions that equal $\pi$. For example

$$\begin{aligned}
\pi &= 2\left(\frac{2\times 2\times 4\times 4\times 6\times 6\times 8\times\ldots}{1\times 3\times 3\times 5\times 5\times 7\times 7\times\ldots}\right)\\
&= 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots\right)\\
&= \sqrt{\frac{6}{1^2}+\frac{6}{2^2}+\frac{6}{3^2}+\frac{6}{4^2}+\ldots}\\
&= 3\left(1+\frac{1^2}{4\times 6}+\frac{1^2\times 3^2}{4\times 6\times 8\times 10}+\frac{1^2\times 3^2\times 5^2}{4\times 6\times 8\times 10\times 12\times 14}+\ldots\right)
\end{aligned}$$

However, I think the most remarkable formula for the computation of $\pi$ has got to be from India. Read on ...

1. Around 1910, the Indian mathematician Srinivasa Ramanujan discover the formula

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26390n)}{(n!)^4\,396^{4n}}.$$

William Gosper (he's from New Jersey) used this series in 1985 to compute the first seventeen million digits of $\pi$.

(a) 5 points Verify that this series is convergent.

**Solution:** Using the **Ratio Test**.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(4n+4)!\,(1103+26390n+26390)}{[(n+1)!]^4\,396^{4n+4}}\cdot\frac{(n!)^4\,396^{4n}}{(4n)!\,(1103+26390n)}\right|\\
&= \lim_{n\to\infty}\left|\frac{(4n+4)\,(4n+3)\,(4n+2)\,(4n+1)\,(27493+26390n)}{(n+1)\,(n+1)\,(n+1)\,(n+1)\,396^4\,(1103+26390n)}\right|\\
&= \frac{4^4}{396^4}<1
\end{aligned}$$

So the series converges by the **Ratio Test**.

(b) 5 points How many correct decimal places of $\pi$ do you get if you just use the first term of the series? What if you use two terms?

**Solution:** 3.141592653589793238462643383280 is what I got when I asked Mathematica to compute $\pi$ to thirty places of accuracy. Using the first term of Ramanujan's series I get:

$$\left[\frac{2\sqrt{2}}{9801}\cdot 1103\right]^{-1} \approx 3.14159273001,$$

which is accurate to the sixth decimal place. Using the first two terms of Ramanujan's series I get:

$$\left[\frac{2\sqrt{2}}{9801}\cdot\left(1103+\frac{4!\,(27493)}{396^4}\right)\right]^{-1} \approx 3.141592653589793878,$$

which is accurate to the fifteen decimal place. Your accuracy may differ, depending on what you're using to calculate.[1] For example, a handheld calculator should be able to do the first term without trouble, but the two-termed expression may differ from what is presented here. However the point is that this is a really good approximation and the series appears to converge very quickly. I guess that's why William Gosper used it.