1. Consider the three infinite series below.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n^2-1)}{4n^3-2n+1}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{5(-4)^{n+2}}{3^{2n+1}}$

(a) Which of these series is (are) alternating?

Solution: Clearly series $\boxed{\text{(i) and (iii)}}$. It is not necessary to expand to see that they are alternating.

(b) Which one of these series diverges, and why?

Solution: Series $\boxed{\text{(ii)}}$. To show this, just show that the limit of the n^{th} term as $n \to \infty$ is not equal to zero.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)(n^2-1)}{4n^3-2n+1} &= \lim_{n\to\infty} \frac{n^3+n^2-n-1}{4n^3-2n+1} \\
&= \lim_{n\to\infty} \frac{1+1/n-1/n^2-1/n^3}{4-2/n^2+1/n^3} = \frac{1}{4} \neq 0
\end{aligned}$$

(c) One of these series converges absolutely. Which one? Compute its sum.

Solution: Series $\boxed{\text{(iii)}}$ coverges absolutely, and its sum is given by the geometric series.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5(-4)^{n+2}}{3^{2n+1}} &= \sum_{n=1}^{\infty} \frac{80(-4)^n}{3 \cdot 9^n} \\
&= \frac{80}{3} \cdot \sum_{n=1}^{\infty} \left(-\frac{4}{9}\right)^n \\
&= \frac{80}{3} \cdot \left[\left(-\frac{4}{9}\right) + \left(-\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^3 + \cdots\right] \\
&= -\frac{320}{27} \cdot \left[1 + \left(-\frac{4}{9}\right) + \left(-\frac{4}{9}\right)^2 + \cdots\right] \\
&= -\frac{320}{27} \cdot \frac{1}{1+4/9} = -\frac{320}{27} \cdot \frac{9}{9+4} = -\frac{320}{27} \cdot \frac{9}{13} = \boxed{-\frac{320}{39}}
\end{aligned}$$

2. Given that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

is a solution to the differential equation

$$f''(x) + f(x) = 0.$$

Answer the following questions.

(a) Find $f'(x)$.

Solution:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ f'(x) &= 0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\ f'(x) &= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}} \end{aligned}$$

(b) Find $f''(x)$.

Solution:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \\ f'(x) &= -\frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\ f''(x) &= -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ f''(x) &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}} \end{aligned}$$

(c) Expand both f and f'' and then substitute into the differential equation

$$f''(x) + f(x) = 0,$$

to verify that it is a solution.

Solution:

$$\begin{aligned}
0 &= f''(x) + f(x) \\
0 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n}}{(2n)!} \\
0 &= \left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
0 &= 0
\end{aligned}$$

3. Given

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, \mathrm{d}x,$$

and the following graph.

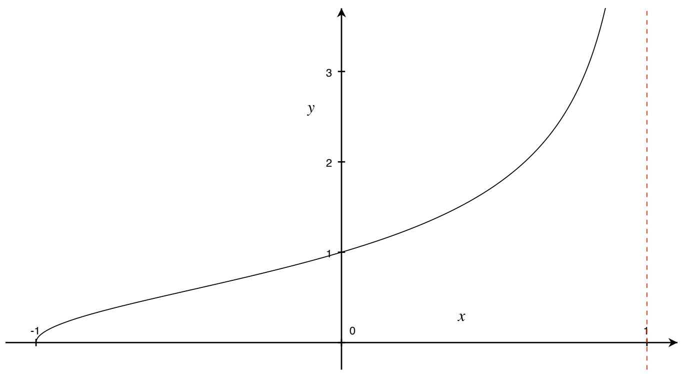

Figure 1: Partial graph of $y = \sqrt{\frac{1+x}{1-x}}$ and $x = 1$.

(a) Why is this an improper integral?

Solution: At $x = 1$ it is undefined.

(b) Show that

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}},$$

if $-1 < x < 1$.

Solution:

$$\sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{1+x}{\sqrt{1-x^2}}$$

Or, if you prefer.

$$\frac{1+x}{\sqrt{1-x^2}} = \frac{1+x}{\sqrt{(1+x)(1-x)}} = \sqrt{\frac{1+x}{1-x}}$$

(c) Use
$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}},$$
to evaluate
$$\int \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x.$$

Solution:
$$\begin{aligned}\int \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x &= \int \frac{1+x}{\sqrt{1-x^2}}\ \mathrm{d}x \\ &= \int \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x + \int \frac{x}{\sqrt{1-x^2}}\ \mathrm{d}x \\ &= \boxed{\arcsin x - \sqrt{1-x^2} + C}\end{aligned}$$

(d) Evaluate
$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x.$$

Solution:
$$\begin{aligned}\lim_{a\to 1^-}\int_{-1}^{a} \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x &= \lim_{a\to 1^-}\left(\arcsin x - \sqrt{1-x^2}\right)\Big]_{-1}^{a} \\ &= \lim_{a\to 1^-}\left[\arcsin a - \sqrt{1-a^2}\right] - \arcsin(-1) \\ &= \arcsin 1 - \arcsin(-1) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}\end{aligned}$$

4. Integrate.
$$\int_{-1}^{1} \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\ \mathrm{d}x.$$

Solution: First you'll need to long divide.
$$\frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \frac{x+5}{x^2-2x-8}.$$
Then use *partial fractions* to get
$$\frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \frac{x+5}{x^2-2x-8} = 2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}.$$

Now integrate.

$$\begin{aligned}\int_{-1}^{1} \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,\mathrm{d}x &= \int_{-1}^{1} 2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\,\mathrm{d}x \\ &= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2|\Big]_{-1}^{1} \\ &= \boxed{\ln\frac{3}{5\sqrt{5}}}\end{aligned}$$

5. Given a differential equation of the form

$$y' = kxy^2,$$

find the constant k such that

$$y = \frac{1}{x^2+5}$$

is a solution to this differential equation.

Solution: First find y'

$$y' = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^2+5}\right] = -\frac{2x}{\left(x^2+5\right)^2},$$

and then see what k is in the differential equation,

$$\begin{aligned} y' &= kxy^2 \\ -\frac{2x}{\left(x^2+5\right)^2} &= kx\left(\frac{1}{x^2+5}\right)^2 \\ k &= \boxed{-2}\end{aligned}$$

6. Use integration by parts to evaluate

$$\int_1^2 x^3 \ln x\,\mathrm{d}x.$$

Solution: Starting with,

$$u = \ln x \qquad \text{and} \qquad \mathrm{d}v = x^3\,\mathrm{d}x,$$

then,

$$du = \frac{1}{x}\,dx \qquad \text{and} \qquad v = \frac{x^4}{4}.$$

Now carefully using these parts we finally have.

$$\begin{aligned}
\int_1^2 x^3 \ln x \, dx &= \left.\frac{x^4 \ln x}{4}\right]_1^2 - \frac{1}{4}\int_1^2 \frac{1}{x}\cdot\frac{x^4}{4}\,dx \\
&= \left.\frac{x^4 \ln x}{4}\right]_1^2 - \frac{1}{4}\int_1^2 x^3\,dx \\
&= \left.\frac{x^4 \ln x}{4} - \frac{x^4}{16}\right]_1^2 \\
&= \boxed{\ln 16 - \frac{15}{16}}
\end{aligned}$$

7. **Euler's identity** is helpful when dealing with complex numbers, it states that for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

You should know from basic algebra how to raise i (where $\sqrt{-1} = i$) to a natural number power. Here's a short list:

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i, \ldots.$$

Let's use our series expansion for e^x, but this time let's replace x by $i\theta$ and see if you can get **Euler's identity** by doing this. Use this identity to find out a simple form (it's a very simple and important number) for $e^{i\pi}$.

Solution:

$$\begin{aligned}
e^{\theta} &= 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \cdots \\
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

Using **Euler's identity** with $\theta = \pi$ we get:

$$\begin{aligned}
e^{i\pi} &= \cos\pi + i\sin\pi \\
e^{i\pi} &= -1 + i\cdot 0 \\
e^{i\pi} &= -1
\end{aligned}$$

This, of course leads to a very important relationship between π, 1, 0, e and i, namely

$$\boxed{e^{i\pi} + 1 = 0}.$$

These five numbers are often referred to as the five most important numbers in mathematics. [Hate to sound biased, but these numbers are really in a class by themselves. In a way, numbers are polytheistic, and these five numbers stand equally above all others.] When I travel I almost always visit cultural centers where people are asked to leave a comment in a guest book, and I often leave $e^{i\pi} + 1 = 0$ because those symbols are universally understood by educated people. Once I saw it crossed off (with a comment stating that it was not true), I guess because it offended someone. Anyway, I just might start leaving $1 = 0.\overline{999}$ which probably appears equally offensive—I know it once offended me when I first saw calculus.

8. Evaluate.[1] You must show work!

$$\int_1^4 \sin\sqrt{x}\;\mathrm{d}x$$

Solution: Taking the hint.

$$u = \sqrt{x} \quad\Rightarrow\quad \mathrm{d}u = \frac{1}{2\sqrt{x}}\;\mathrm{d}x \quad\Rightarrow\quad \mathrm{d}u = \frac{1}{2u}\;\mathrm{d}x \quad\Rightarrow\quad 2u\;\mathrm{d}u = \mathrm{d}x$$

Here's goes.

$$\int_1^4 \sin\sqrt{x}\;\mathrm{d}x = 2\int_1^2 u\sin u\;\mathrm{d}u$$

Now here's the parts.

$$v = u \quad\Rightarrow\quad \mathrm{d}v = \mathrm{d}u \quad\text{and}\quad \sin u\;\mathrm{d}u = \mathrm{d}w \quad\Rightarrow\quad -\cos u = w$$

Here goes.

$$\begin{aligned}
\int_1^4 \sin\sqrt{x}\;\mathrm{d}x &= 2\int_1^2 u\sin u\;\mathrm{d}u \\
&= -2u\cos u\;\Big]_1^2 + 2\int_1^2 \cos u\;\mathrm{d}u \\
&= -2u\cos u + 2\sin u\;\Big]_1^2 \\
&= \boxed{-4\cos 2 + 2\sin 2 + 2\cos 1 - 2\sin 1}
\end{aligned}$$

[1] First make a simple u-substitution where $u = \sqrt{x}$, then use integration by parts.

9. The **generalized Binomial Theorem** was discovered by Isaac Newton around 1665, and you probably learned the **Binomial Theorem** in pre-calculus. It was probably introduced as an expansion of $(a+b)^n$.

$$(a+b)^n = a^n + k_1 a^{n-1} b + k_2 a^{n-2} b^2 + k_3 a^{n-3} b^3 + \cdots + k_{n-2} a^2 b^{n-2} + k_{n-1} a b^{n-1} + b^n.$$

The pattern should be easy to follow, but the constants k_i are in fact difficult to compute. After some thought though, most students can figure out that these coefficients form a pattern for $n \in \mathbb{Z}^+$. As you may recall, the coefficients of the binomial expansion can be computed using this formula:

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!},$$

where n represents the degree (row in Pascal's Triangle) and r represents the position starting with 0 and ending with n in each expansion. The **generalized Binomial Theorem** does not limit the power to being an integer though. We will be using the Taylor series to develop the **generalized Binomial Theorem** which states for any exponent $a \in \mathbb{R}$, integer $n \geq 0$, and $|x| < 1$:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n} x^n + \cdots$$

Here we need to define the binomial coefficient

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}.$$

For example

$$\binom{4/3}{3} = \frac{4/3 \cdot (4/3-1)(4/3-2)}{3!} = -\frac{4}{81}.$$

So now, with what you know about Taylor series, try to develop the **generalized Binomial Theorem**.

(a) Repeatedly take derivatives of

$$f(x) = (1+x)^a,$$

and try to find a simple pattern for the n^{th} derivative.

Solution:

$$\begin{aligned} f(x) &= (1+x)^a \\ f'(x) &= a(1+x)^{a-1} \\ f''(x) &= a(a-1)(1+x)^{a-2} \\ f'''(x) &= a(a-1)(a-2)(1+x)^{a-3} \\ \vdots &= \vdots \\ f^{(n)}(x) &= \boxed{a(a-1)(a-2)\cdots(a-n+1)(1+x)^{a-n}} \end{aligned}$$

(b) Now we need to evaluate these derivatives at $x = 0$, and derive the **Taylor series** using this information.

Solution:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= a \\ f''(0) &= a(a-1) \\ f'''(0) &= a(a-1)(a-2) \\ \vdots &= \vdots \\ f^{(n)}(0) &= a(a-1)(a-2)\cdots(a-n+1) \end{aligned}$$

So the general coefficient of the **Taylor series** is

$$\frac{f^{(n)}(0)}{n!} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

The result being:

$$\begin{aligned} (1+x)^a &= \boxed{1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \end{aligned}$$

10. Find the area of the region bounded by the given curves.

$$y = xe^{-0.4x}, \quad y = 0, \quad x = 5$$

Solution: I am using integration by parts with $u = x$ and $e^{-0.4x}\ \mathrm{d}x = \mathrm{d}v$.

$$\begin{aligned} \int_0^5 xe^{-0.4x}\ \mathrm{d}x &= -\frac{5x}{2e^{0.4x}}\Big]_0^5 + \frac{5}{2}\int_0^5 e^{-0.4x}\ \mathrm{d}x \\ &= -\frac{5x}{2e^{0.4x}} - \frac{25}{4e^{0.4x}}\Big]_0^5 \\ &= -\frac{25}{2e^2} - \frac{25}{4e^2} + \frac{25}{4} \\ &= \boxed{\frac{25e^2 - 75}{4e^2}} \end{aligned}$$

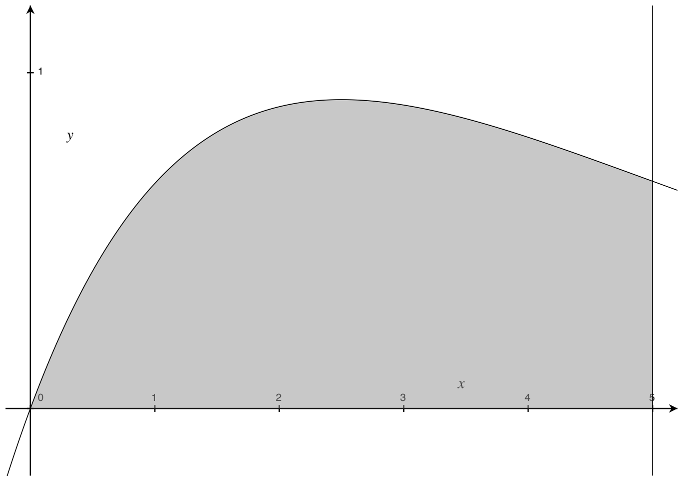

Figure 2: Area of interest.

Here's the graph.

11. Make a substitution to express the integral as a rational function and then evaluate the integral.

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\ \mathrm{d}x$$

Solution: Let $u^2 = x$, and for $x > 0$ we also have $u = \sqrt{x}$, furthermore $2u\ \mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}
\int_9^{16} \frac{\sqrt{x}}{x-4}\ \mathrm{d}x &= \int_3^4 \frac{2u^2}{u^2-4}\ \mathrm{d}u \\
&= \int_3^4 2 + \frac{8}{u^2-4}\ \mathrm{d}u \\
&= \int_3^4 2 - \frac{2}{u+2} + \frac{2}{u-2}\ \mathrm{d}u \\
&= 2u - 2\ln|u+2| + 2\ln|u-2| \quad \Big]_3^4 \\
&= 2u + 2\ln\left|\frac{u-2}{u+2}\right|\Big]_3^4 \\
&= \left[8 + \ln\left(\frac{1}{9}\right)\right] - \left[6 + \ln\left(\frac{1}{25}\right)\right] \\
&= \boxed{2 + \ln\left(\frac{25}{9}\right)}
\end{aligned}$$

12. Evaluate the integral.

$$\int_0^1 \frac{y}{e^{2y}}\,\mathrm{d}y$$

Solution: Using integration by parts with $u = y$ and $e^{-2y}\,\mathrm{d}y = \mathrm{d}v$.

$$\begin{aligned}
\int_0^1 \frac{y}{e^{2y}}\,\mathrm{d}y &= \int_0^1 ye^{-2y}\,\mathrm{d}y \\
&= -\frac{y}{2e^{2y}}\Big]_0^1 + \frac{1}{2}\int_0^1 e^{-2y}\,\mathrm{d}y \\
&= -\frac{y}{2e^{2y}} - \frac{1}{4e^{2y}}\Big]_0^1 \\
&= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} \\
&= \boxed{\frac{e^2-3}{4e^2}}
\end{aligned}$$

13. Given that

$$f(x) = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}.$$

Try to find a power series for $f(x)$ and its interval of convergence.

Solution: You should note that we'll be using

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

and

$$\begin{aligned}
\frac{2}{1-2x} = 2\left[\frac{1}{1-(2x)}\right] &= 2\left[1 + 2x + 4x^2 + 8x^3 + \cdots + (2x)^n + \cdots\right] \\
&= 2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots
\end{aligned}$$

so

$$\begin{aligned}
\frac{2}{1-2x} - \frac{1}{1-x} &= \left[2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots\right] - \left[1 + x + x^2 + \cdots\right] \\
&= 1 + 3x + 7x^2 + 15x^3 + \cdots + \left(2^{n+1}-1\right)x^n + \cdots \\
&= \sum_{n=0}^{\infty}\left(2^{n+1}-1\right)x^n
\end{aligned}$$

Using the **Ratio Test** the interval of convergence is $(-1/2,\ 1/2)$.

14. Evaluate (exact answer) without using a computer.

$$\int_{\pi/4}^{\pi/2} \cot^3 x \, \mathrm{d}x$$

Solution: On line (4) I am using $u = \csc x$ on the first integral and $u = \sin x$ on the second integral.

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cot^3 x \, \mathrm{d}x &= \int_{\pi/4}^{\pi/2} \cot x \left(\cot^2 x\right) \, \mathrm{d}x && (1)\\
&= \int_{\pi/4}^{\pi/2} \cot x \left(\csc^2 x - 1\right) \, \mathrm{d}x && (2)\\
&= \int_{\pi/4}^{\pi/2} \cot x \; \csc^2 x \, \mathrm{d}x - \int_{\pi/4}^{\pi/2} \cot x \, \mathrm{d}x && (3)\\
&= \int_{\pi/4}^{\pi/2} \cot x \; \csc x \; \csc x \, \mathrm{d}x - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, \mathrm{d}x && (4)\\
&= \left(-\frac{1}{2}\csc^2 x - \ln|\sin x|\right)\Bigg]_{\pi/4}^{\pi/2} && (5)\\
&= \boxed{\frac{1 - \ln 2}{2}} && (6)
\end{aligned}$$

15. Find the sum of

$$\sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k}.$$

Solution: Expanding, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k} &= e^{-k}\left[1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \frac{k^5}{5!} + \frac{k^6}{6!} + \cdots\right]\\
&= e^{-k}\left[e^k\right] = \boxed{1}
\end{aligned}$$

16. Evaluate the Integral.

$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x$$

Solution: This is an irreducible quadratic factor. Let's work on the rational expression first.

$$\begin{aligned}\frac{x+4}{x^2+2x+5} &= \frac{1}{2}\cdot\frac{2x+2}{x^2+2x+5}+\frac{3}{x^2+2x+5}\\ &= \frac{1}{2}\frac{2x+2}{x^2+2x+5}+\frac{3}{(x+1)^2+4}\\ &= \frac{1}{2}\cdot\frac{2x+2}{x^2+2x+5}+\frac{3}{4}\cdot\frac{1}{[(x+1)/2]^2+1}\end{aligned}$$

So the integration becomes:

$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x = \frac{1}{2}\cdot\int\frac{2x+2}{x^2+2x+5}\ \mathrm{d}x+\frac{3}{4}\cdot\int\frac{1}{[(x+1)/2]^2+1}\ \mathrm{d}x$$

Now let's do one integration at a time. For

$$\frac{1}{2}\cdot\int\frac{2x+2}{x^2+2x+5}\ \mathrm{d}x,$$

let $u = x^2+2x+5$ and then $\mathrm{d}u = (2x+2)\ \mathrm{d}x$, resulting in:

$$\frac{1}{2}\cdot\int\frac{2x+2}{x^2+2x+5}\ \mathrm{d}x = \frac{1}{2}\cdot\int^{*}\frac{1}{u}\ \mathrm{d}u = \frac{1}{2}\ln|u|+C_1 = \ln\sqrt{x^2+2x+5}+C_1.$$

The second integration

$$\frac{3}{4}\cdot\int\frac{1}{[(x+1)/2]^2+1}\ \mathrm{d}x,$$

requires that we let $u = (x+1)/2$ and then $2\mathrm{d}u = \mathrm{d}x$, resulting in:

$$\frac{3}{4}\cdot\int\frac{1}{[(x+1)/2]^2+1}\ \mathrm{d}x = \frac{3}{2}\cdot\int\frac{1}{u^2+1}\ \mathrm{d}u = \arctan u + C_2 = \frac{3}{2}\arctan\left(\frac{x+1}{2}\right)+C_2.$$

Combining the two we finally have:

$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x = \boxed{\ln\sqrt{x^2+2x+5}+\frac{3}{2}\arctan\left(\frac{x+1}{2}\right)+C}$$

17. Find the radius of convergence and the interval of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}.$$

Solution: Using the **Ratio Test**, we have

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!x^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+1)\,x}{(2n+1)}\right| \\ &= \frac{|x|}{2} \end{aligned}$$

So by the **Ratio Test**, the series converges for $|x| < 2$, so the $\boxed{\text{radius of convergence is } 2}$. To find the interval of convergence we need to test the endpoints ± 2. When $x = 2$ we get

$$\begin{aligned} a_n &= \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n \cdot 2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \left(\frac{2}{1}\right) \cdot \left(\frac{4}{3}\right) \cdot \left(\frac{6}{5}\right) \cdot \left(\frac{8}{7}\right) \cdot \cdots \cdot \left(\frac{2n}{2n-1}\right) \end{aligned}$$

I think it is *obvious* that $a_n > 1$ for all n. Now if we were to use $x = -2$ we can still state that $|a_n| > 1$ for all n. So we have

$$\lim_{n\to\infty} |a_n| \neq 0,$$

and using the **Test for Divergence** we can state that the series diverges for $x = \pm 2$, so the $\boxed{\text{interval of convergence is } (-2,\ 2)}$.

18. For what values of p is the series convergent?

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$$

Solution: First off, if $p = 0$ we have an alternating harmonic series which is convergent. If $p < 0$ we clearly have the a_n's decreasing as n increases. If $p > 0$ we need to let

$$f(x) = \frac{(\ln x)^p}{x},$$

and its derivative is

$$f'(x) = \frac{(\ln x)^{(p-1)} (p - \ln x)}{x^2}.$$

Now, if $x > e^p$, then $f'(x) < 0$. So, for $n \geq \lceil e^p \rceil$,

$$\left| (-1)^{n-1} \frac{(\ln n)^p}{n} \right|$$

is decreasing. Using the **Alternating Series Test**, we can state that

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$$

is $\boxed{\text{convergent for all } p}$.

19. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

Solution: Using the Ratio Test, we have

$$\begin{aligned}
\lim_{n \to infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \frac{(n+1)!\,(n+1)!}{(kn+k)!} \div \frac{(n!)^2}{(kn)!} \\
&= \lim_{n \to \infty} \frac{(n+1)!\,(n+1)!}{(kn+k)!} \cdot \frac{(kn)!}{(n!)^2} \\
&= \lim_{n \to \infty} \frac{(n+1)^2}{(kn+k)(kn+k-1)(kn+k-2)\cdot \cdots \cdot (kn+1)}
\end{aligned}$$

Okay, if $k = 1$ we have

$$\lim_{n \to \infty} \frac{(n+1)^2}{(n+1)} = \infty,$$

so the series diverges. If $k = 2$ we have

$$\lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

so the series converges. If $k > 2$ the degree of the denominator increases and the limit is zero. So the series converges for $\boxed{k \geq 2, \text{ where } k \in \mathbb{Z}}$.

20. Using what you already know[2] about the Taylor series for e^x.

(a) Find the Taylor series for

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Solution: You should know, or be able to derive that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

and is true for all x. Hence, using this expansion for e^x you should be able to simply derive that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

for all x. Now adding e^x and e^{-x} together, we get

$$e^x + e^{-x} = 2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + 2\cdot\frac{x^6}{6!} + \cdots.$$

Finally, dividing by 2, we have

$$\cosh x = \frac{e^x + e^{-x}}{2} = \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}.$$

(b) Looking at the Taylor series for $\cosh x$, explain why it looks like a parabola near $x = 0$. What is the equation of this parabola? Graph both $\cosh x$ and the parabola to see if it's a good fit near zero.

Solution: For small x, the $\boxed{\text{increasing degree terms play a minor role near zero}}$, however, if we $\boxed{\text{include the second degree term}}$ we can see the parabolic nature of $\cosh x$. The parabola that best fits $\cosh x$ near zero is

$$\boxed{y = 1 + \frac{x^2}{2!}}.$$

[2]Please do not take derivatives.

Here's the partial graph.

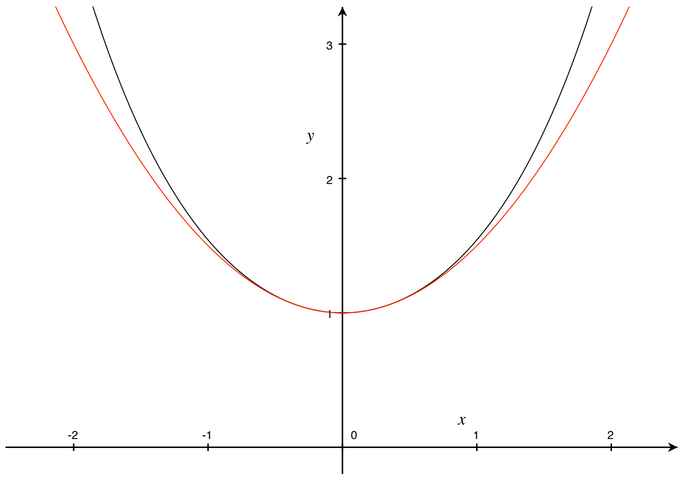

Figure 3: Partial graph of both $y = \cosh x$ (in black) and $y = 1 + x^2/2$ (in red).

21. By looking at the Taylor series, decide which of the folowing functions is largest, and which is smallest, for small positive θ.

$$1 + \sin\theta, \qquad \cos\theta, \qquad \frac{1}{1-\theta^2}$$

Solution: Using what we already know. For example, using what you learned in pre-calculus, especially about non-linear inequalities, might be insightful when looking at these expansions.

$$\begin{aligned} 1 + \sin\theta &= 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \cdots \\ \frac{1}{1-\theta^2} &= 1 + \theta^2 + \theta^4 + \theta^6 + \theta^{10} + \cdots \end{aligned}$$

Clearly $\boxed{1 + \sin\theta \text{ is the largest}}$ amongst the three series when θ is positive and near zero. Now looking at the remaining two series it is clear that $\boxed{\cos\theta \text{ is smallest}}$. Here's a partial graph with the viewing window deliberately set to $x \in [-0.0078,\ 0.3047]$ and $y \in [0.8945,\ 1.1117]$. Again, please make every effort to learn how to use technology to help visualize complicated graphs.

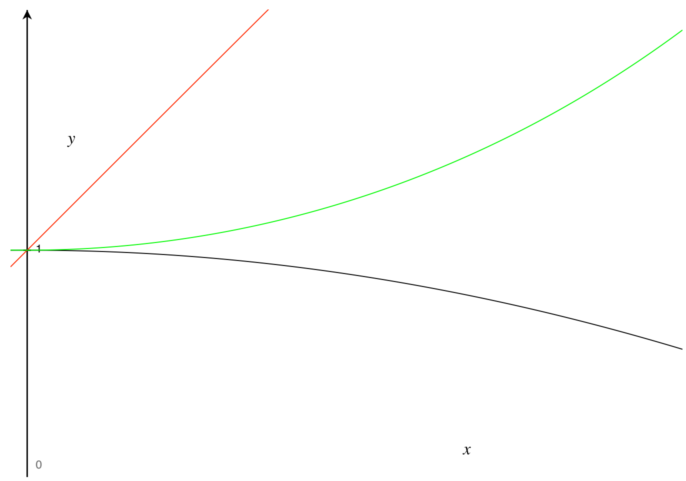

Figure 4: $y = 1 + \sin x$ in red; $y = \cos x$ in black; $y = (1 - x^2)^{-1}$ in green.

22. Use a *known* series to evaluate

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Solution:

$$\begin{aligned} \frac{\sin x - x}{x^3} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - x}{x^3} \\ &= -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots \\ \lim_{x \to 0} \frac{\sin x - x}{x^3} &= \lim_{x \to 0} \left[-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots \right] = \boxed{-\frac{1}{6}} \end{aligned}$$

23. Calculate

$$\frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots$$

Solution: This is the series for sine evaluated at $\pi/2$.

$$\sin \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \cdots = \boxed{1}$$

24. Suppose you were given an *infinite* sequence of circles with the following radii:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots ,$$

what is the total area of these circles?

Solution: This is an infinite geometric sum

$$\begin{aligned}
\pi\left(\frac{1}{2}\right)^2 + \pi\left(\frac{1}{4}\right)^2 + \pi\left(\frac{1}{8}\right)^2 + \cdots &= \pi\left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right] \\
&= \pi \cdot \frac{1/4}{1-1/4} = \boxed{\frac{\pi}{3}}
\end{aligned}$$

25. Find the sum of

$$\sum_{n=1}^{\infty} n x^{n-1}$$

for $|x| < 1$.

Solution: Okay, this may be a tough one, especially if you were taking partial sums and looking for patterns. However, this looks like it might be related to the following Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

To see this relationship, take the derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + x + x^2 + x^3 + x^4 + \cdots\right] \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\boxed{\frac{1}{(1-x)^2}} &= \sum_{n=1}^{\infty} n x^{n-1}
\end{aligned}$$

26. Consider the loop of the curve defined by

$$6y^2 = x\left(2-x\right)^2 .$$

A graph is provided. Find the area of the surface generated by rotating this loop about:

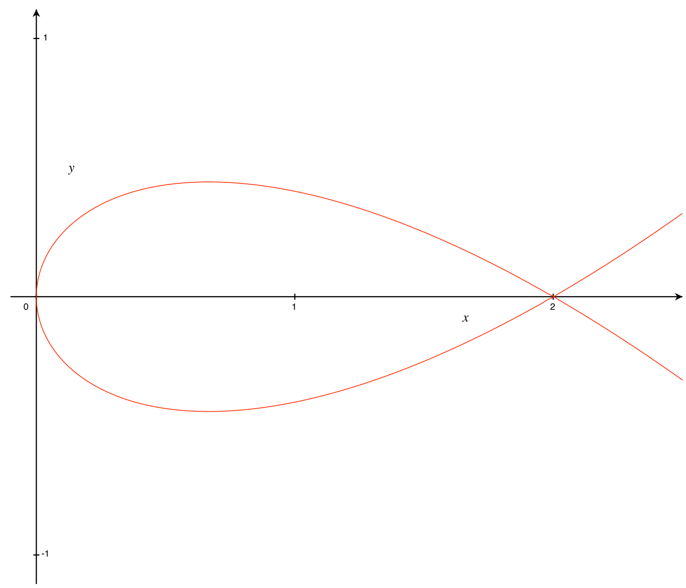

Figure 5: Partial graph of $6y^2 = x\left(2-x\right)^2$.

(a) x-axis;

Solution:

$$\begin{aligned}
\int_0^2 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x &= \int_0^2 2\pi\left[\frac{(2-x)\sqrt{x}}{\sqrt{6}}\right]\sqrt{1+\left(\frac{2-3x}{2\sqrt{6x}}\right)^2}\ \mathrm{d}x \\
&= \int_0^2 2\pi\left[\frac{(2-x)\sqrt{x}}{\sqrt{6}}\right]\sqrt{\frac{9x^2+12x+4}{24x}}\ \mathrm{d}x \\
&= \int_0^2 2\pi\left[\frac{(2-x)\sqrt{x}}{\sqrt{6}}\right]\sqrt{\frac{(3x+2)^2}{24x}}\ \mathrm{d}x \\
&= \frac{\pi}{6}\int_0^2 (2-x)(3x+2)\ \mathrm{d}x \\
&= \frac{\pi}{6}\int_0^2 4+4x-3x^2\ \mathrm{d}x \\
&= \frac{\pi}{6}\left(4x+2x^2-x^3\right)\Big]_0^2 = \boxed{\frac{4\pi}{3}}
\end{aligned}$$

(b) y-axis.

Solution:

$$\begin{aligned}
2\int_0^2 2\pi x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= 2\int_0^2 2\pi x\sqrt{1+\left(\frac{2-3x}{2\sqrt{6x}}\right)^2}\,\mathrm{d}x \\
&= 2\int_0^2 2\pi x\sqrt{\frac{9x^2+12x+4}{24x}}\,\mathrm{d}x \\
&= 2\int_0^2 2\pi x\sqrt{\frac{(3x+2)^2}{24x}}\,\mathrm{d}x \\
&= \frac{2\pi}{\sqrt{6}}\int_0^2 \sqrt{x}\,(3x+2)\,\mathrm{d}x \\
&= \frac{2\pi}{\sqrt{6}}\left(\frac{x^{3/2}\,(18x+20)}{15}\right)\Bigg]_0^2 = \boxed{\frac{224\pi}{15\sqrt{3}}} = \frac{224\pi\sqrt{3}}{45}
\end{aligned}$$

27. (a) Use differentiation of a *known* power series to find the power series representation for

$$f(x) = \frac{1}{(1+x)^2}.$$

What is the radius of convergence?

Solution: You should note that we'll be using

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

to start this problem off. A bit *tricky*, but here goes . . . .

$$\begin{aligned}
f(x) = \frac{1}{(1+x)^2} &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-(-x)}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left[1 - x + x^2 - x^3 + \cdots\right] \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots \\
&= \boxed{\sum_{n=0}^{\infty}(-1)^n (n+1)\,x^n}
\end{aligned}$$

You can use the **Ratio Test** to show that the $\boxed{\text{radius of convergence is } R = 1}$. By the way, the interval of convergence is $(-1,\ 1)$.

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}.$$

What is the radius of convergence?

Solution:

$$\begin{aligned} f(x) = \frac{1}{(1+x)^3} &= -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(1+x)^2}\right] \\ &= -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} (-1)^n (n+1) x^n\right] \\ &= -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[1 - 2x + 3x^2 - 4x^3 + \cdots\right] \\ &= 1 - 3x + 6x^2 - 10x^3 + \cdots \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n} \end{aligned}$$

You can use the **Ratio Test** to show that the $\boxed{\text{radius of convergence is } R = 1}$. By the way, the interval of convergence is $(-1,\ 1)$.

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}.$$

What is the radius of convergence?

Solution:

$$\begin{aligned} f(x) = \frac{x^2}{(1+x)^3} &= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n \\ &= x^2\left[1 - 3x + 6x^2 - 10x^3 + \cdots\right] \\ &= \left[x^2 - 3x^3 + 6x^4 - 10x^5 + \cdots\right] \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^{n+2}} \end{aligned}$$

You can use the **Ratio Test** to show that the $\boxed{\text{radius of convergence is } R = 1}$. By the way, the interval of convergence is $(-1,\ 1)$.

28. Consider the initial value problem

$$\frac{dy}{dx} = \frac{x\left(1+y^2\right)}{2}, \quad y(0) = 1.$$

Sketch the solution to this initial value problem, and use your sketch to estimate $y(1)$. Also, given that

$$y(x) = \tan\left(\frac{x^2}{4} + \frac{\pi}{4}\right)$$

is a solution to this differential equation, estimate the true value of $y(1)$.

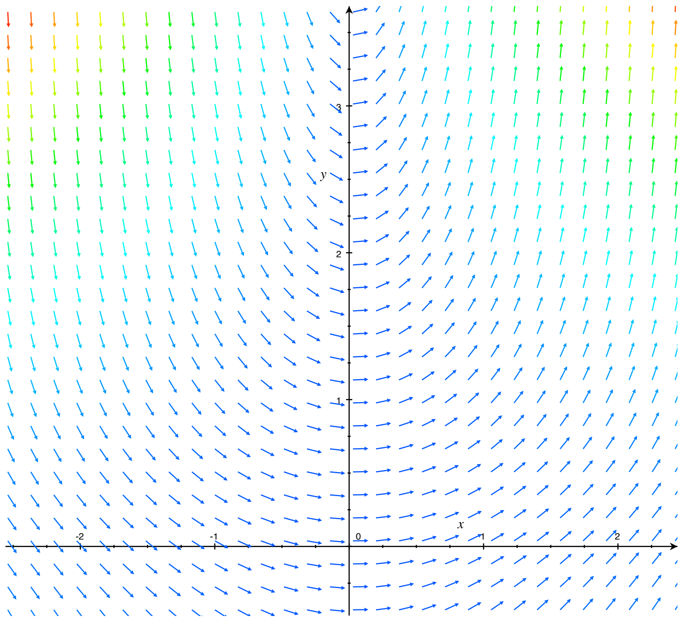

Figure 6: Direction field.

Solution: My sketch may be better than yours (I'm using software). From my graph I get $\boxed{y(1) = 1.7}$. Using the formula provided,

$$y(1) = \tan\left(\frac{1}{4} + \frac{\pi}{4}\right) \approx \boxed{1.68579641717}.$$

Not bad.

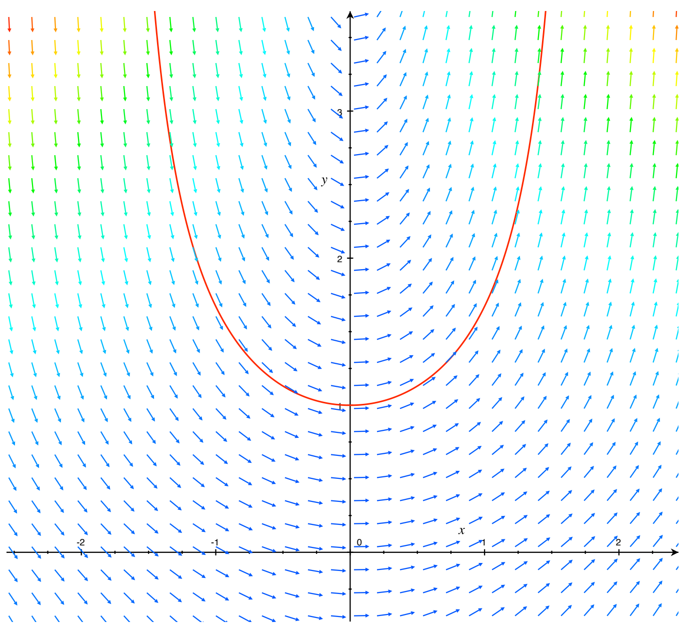

Figure 7: Direction field with solution.