**Name:** ________________________________

**Signature:** ________________________________

Show all work clearly and in order, and box your final answers. Justify your answers whenever possible. You have 80 minutes to take this 100 point exam.

1. 20 points Find the area of the region bounded by the given curves.

$$y = xe^{-0.4x}, \quad y = 0, \quad x = 5$$

**Solution:** I am using integration by parts with $u = x$ and $e^{-0.4x}\ \mathrm{d}x = \mathrm{d}v$.

$$\begin{aligned}
\int_0^5 xe^{-0.4x}\ \mathrm{d}x &= -\frac{5x}{2e^{0.4x}}\Bigg]_0^5 + \frac{5}{2}\int_0^5 e^{-0.4x}\ \mathrm{d}x \\
&= -\frac{5x}{2e^{0.4x}} - \frac{25}{4e^{0.4x}}\Bigg]_0^5 \\
&= -\frac{25}{2e^2} - \frac{25}{4e^2} + \frac{25}{4} \\
&= \boxed{\frac{25e^2 - 75}{4e^2}}
\end{aligned}$$

Here's the graph.

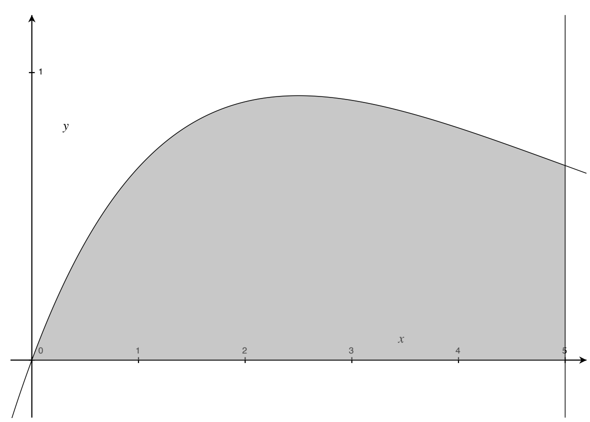


Figure 1: Area of interest.

2. [20 points] Make a substitution to express the integral as a rational function and then evaluate the integral.

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\ \mathrm{d}x$$

**Solution:** Let $u^2 = x$, and for $x > 0$ we also have $u = \sqrt{x}$, furthermore $2u\ \mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}
\int_9^{16} \frac{\sqrt{x}}{x-4}\ \mathrm{d}x &= \int_3^4 \frac{2u^2}{u^2-4}\ \mathrm{d}u \\
&= \int_3^4 2 + \frac{8}{u^2-4}\ \mathrm{d}u \\
&= \int_3^4 2 - \frac{2}{u+2} + \frac{2}{u-2}\ \mathrm{d}u \\
&= 2u - 2\ln|u+2| + 2\ln|u-2| \Big]_3^4 \\
&= 2u + 2\ln\left|\frac{u-2}{u+2}\right| \Big]_3^4 \\
&= \left[8 + \ln\left(\frac{1}{9}\right)\right] - \left[6 + \ln\left(\frac{1}{25}\right)\right] \\
&= \boxed{2 + \ln\left(\frac{25}{9}\right)}
\end{aligned}$$

3. [20 points] Evaluate the integral.

$$\int_0^1 \frac{y}{e^{2y}}\ \mathrm{d}y$$

**Solution:** Using integration by parts with $u = y$ and $e^{-2y}\ \mathrm{d}y = \mathrm{d}v$.

$$\begin{aligned}
\int_0^1 \frac{y}{e^{2y}}\ \mathrm{d}y &= \int_0^1 ye^{-2y}\ \mathrm{d}y \\
&= -\frac{y}{2e^{2y}}\Big]_0^1 + \frac{1}{2}\int_0^1 e^{-2y}\ \mathrm{d}y \\
&= -\frac{y}{2e^{2y}} - \frac{1}{4e^{2y}}\Big]_0^1 \\
&= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} \\
&= \boxed{\frac{e^2-3}{4e^2}}
\end{aligned}$$

4. [20 points] Evaluate (exact answer) without using a computer.

$$\int_{\pi/4}^{\pi/2} \cot^3 x \, \mathrm{d}x$$

**Solution:** On line (4) I am using $u = \csc x$ on the first integral and $u = \sin x$ on the second integral.

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cot^3 x \, \mathrm{d}x &= \int_{\pi/4}^{\pi/2} \cot x \left(\cot^2 x\right) \, \mathrm{d}x && (1)\\
&= \int_{\pi/4}^{\pi/2} \cot x \left(\csc^2 x - 1\right) \, \mathrm{d}x && (2)\\
&= \int_{\pi/4}^{\pi/2} \cot x \ \csc^2 x \, \mathrm{d}x - \int_{\pi/4}^{\pi/2} \cot x \, \mathrm{d}x && (3)\\
&= \int_{\pi/4}^{\pi/2} \cot x \ \csc x \ \csc x \, \mathrm{d}x - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \, \mathrm{d}x && (4)\\
&= \left(-\frac{1}{2}\csc^2 x - \ln|\sin x|\right)\Bigg]_{\pi/4}^{\pi/2} && (5)\\
&= \boxed{\frac{1 - \ln 2}{2}} && (6)
\end{aligned}$$

5. [20 points] Evaluate the Integral.

$$\int \frac{x+4}{x^2+2x+5} \, \mathrm{d}x$$

**Solution:** This is an irreducible quadratic factor. Let's work on the rational expression first.

$$\begin{aligned}
\frac{x+4}{x^2+2x+5} &= \frac{1}{2} \cdot \frac{2x+2}{x^2+2x+5} + \frac{3}{x^2+2x+5}\\
&= \frac{1}{2}\frac{2x+2}{x^2+2x+5} + \frac{3}{(x+1)^2+4}\\
&= \frac{1}{2} \cdot \frac{2x+2}{x^2+2x+5} + \frac{3}{4} \cdot \frac{1}{\left[(x+1)/2\right]^2+1}
\end{aligned}$$

So the integration becomes:

$$\int \frac{x+4}{x^2+2x+5} \, \mathrm{d}x = \frac{1}{2} \cdot \int \frac{2x+2}{x^2+2x+5} \, \mathrm{d}x + \frac{3}{4} \cdot \int \frac{1}{\left[(x+1)/2\right]^2+1} \, \mathrm{d}x$$

Now let's do one integration at a time. For

$$\frac{1}{2} \cdot \int \frac{2x+2}{x^2+2x+5}\ \mathrm{d}x,$$

let $u = x^2 + 2x + 5$ and then $\mathrm{d}u = (2x+2)\ \mathrm{d}x$, resulting in:

$$\frac{1}{2} \cdot \int \frac{2x+2}{x^2+2x+5}\ \mathrm{d}x = \frac{1}{2} \cdot \int^{*} \frac{1}{u}\ \mathrm{d}u = \frac{1}{2} \ln|u| + C_1 = \ln\sqrt{x^2+2x+5} + C_1.$$

The second integration

$$\frac{3}{4} \cdot \int \frac{1}{\left[(x+1)/2\right]^2 + 1}\ \mathrm{d}x,$$

requires that we let $u = (x+1)/2$ and then $2\mathrm{d}u = \mathrm{d}x$, resulting in:

$$\frac{3}{4} \cdot \int \frac{1}{\left[(x+1)/2\right]^2 + 1}\ \mathrm{d}x = \frac{3}{2} \cdot \int \frac{1}{u^2+1}\ \mathrm{d}u = \arctan u + C_2 = \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C_2.$$

Combining the two we finally have:

$$\int \frac{x+4}{x^2+2x+5}\ \mathrm{d}x = \boxed{\ln\sqrt{x^2+2x+5} + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C}$$