Essex County College
Division of Mathematics

Name: ______________________________

Signature: ______________________________

Show all work clearly and in order, and box your final answers. Justify your answers whenever possible. You have 80 minutes to take this 100 point exam.

1. Consider the three infinite series below.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n^2-1)}{4n^3-2n+1}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{5(-4)^{n+2}}{3^{2n+1}}$

(a) $\boxed{\text{10 points}}$ Which of these series is (are) alternating?

Solution: Clearly series $\boxed{\text{(i) and (iii)}}$. It is not necessary to expand to see that they are alternating.

(b) $\boxed{\text{10 points}}$ Which one of these series diverges, and why?

Solution: Series $\boxed{\text{(ii)}}$. To show this, just show that the limit of the n^{th} term as $n \to \infty$ is not equal to zero.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)(n^2-1)}{4n^3-2n+1} &= \lim_{n\to\infty} \frac{n^3+n^2-n-1}{4n^3-2n+1} \\
&= \lim_{n\to\infty} \frac{1+1/n-1/n^2-1/n^3}{4-2/n^2+1/n^3} = \frac{1}{4} \neq 0
\end{aligned}$$

(c) $\boxed{\text{10 points}}$ One of these series converges absolutely. Which one? Compute its sum.

Solution: Series $\boxed{\text{(iii)}}$ coverges absolutely, and its sum is given by the geometric series.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5(-4)^{n+2}}{3^{2n+1}} &= \sum_{n=1}^{\infty} \frac{80(-4)^n}{3\cdot 9^n} \\
&= \frac{80}{3}\cdot\sum_{n=1}^{\infty}\left(-\frac{4}{9}\right)^n \\
&= \frac{80}{3}\cdot\left[\left(-\frac{4}{9}\right)+\left(-\frac{4}{9}\right)^2+\left(-\frac{4}{9}\right)^3+\cdots\right] \\
&= -\frac{320}{27}\cdot\left[1+\left(-\frac{4}{9}\right)+\left(-\frac{4}{9}\right)^2+\cdots\right] \\
&= -\frac{320}{27}\cdot\frac{1}{1+4/9} = -\frac{320}{27}\cdot\frac{9}{9+4} = -\frac{320}{27}\cdot\frac{9}{13} = \boxed{-\frac{320}{39}}
\end{aligned}$$

2. $\boxed{\text{10 points}}$ For what values of p is the series convergent?

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$$

Solution: First off, if $p = 0$ we have an alternating harmonic series which is convergent. If $p < 0$ we clearly have the a_n's decreasing as n increases. If $p > 0$ we need to let

$$f(x) = \frac{(\ln x)^p}{x},$$

and its derivative is

$$f'(x) = \frac{(\ln x)^{(p-1)} (p - \ln x)}{x^2}.$$

Now, if $x > e^p$, then $f'(x) < 0$. So, for $n \geq \lceil e^p \rceil$,

$$\left| (-1)^{n-1} \frac{(\ln n)^p}{n} \right|$$

is decreasing. Using the **Alternating Series Test**, we can state that

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$$

is $\boxed{\text{convergent for all } p}$.

3. $\boxed{\text{10 points}}$ Find the sum of

$$\sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k}.$$

Solution: Expanding, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k} &= e^{-k} \left[1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \frac{k^5}{5!} + \frac{k^6}{6!} + \cdots \right] \\ &= e^{-k} \left[e^k \right] = \boxed{1} \end{aligned}$$

4. $\boxed{\text{10 points}}$ For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

Solution: Using the Ratio Test, we have

$$\begin{aligned}
\lim_{n\to infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!\,(n+1)!}{(kn+k)!}\div\frac{(n!)^2}{(kn)!} \\
&= \lim_{n\to\infty}\frac{(n+1)!\,(n+1)!}{(kn+k)!}\cdot\frac{(kn)!}{(n!)^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(kn+k)\,(kn+k-1)\,(kn+k-2)\cdot\dots\cdot(kn+1)}
\end{aligned}$$

Okay, if $k=1$ we have

$$\lim_{n\to\infty}\frac{(n+1)^2}{(n+1)}=\infty,$$

so the series diverges. If $k=2$ we have

$$\lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)\,(2n+1)}=\frac{1}{4}<1$$

so the series converges. If $k>2$ the degree of the denominator increases and the limit is zero. So the series converges for $\boxed{k\geq 2,\text{ where } k\in\mathbb{Z}}$.

5. Using what you already know[1] about the Taylor series for e^x.

   (a) $\boxed{\text{10 points}}$ Find the Taylor series for

$$\cosh x=\frac{e^x+e^{-x}}{2}.$$

Solution: You should know, or be able to derive that

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

and is true for all x. Hence, using this expansion for e^x you should be able to simply derive that

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$$

for all x. Now adding e^x and e^{-x} together, we get

$$e^x+e^{-x}=2+2\cdot\frac{x^2}{2!}+2\cdot\frac{x^4}{4!}+2\cdot\frac{x^6}{6!}+\cdots.$$

Finally, dividing by 2, we have

$$\cosh x=\frac{e^x+e^{-x}}{2}=\boxed{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots}.$$

[1] Please do not take derivatives.

(b) $\boxed{\text{10 points}}$ Looking at the Taylor series for $\cosh x$, explain why it looks like a parabola near $x = 0$. What is the equation of this parabola? Graph both $\cosh x$ and the parabola to see if it's a good fit near zero.

Solution: For small x, the $\boxed{\text{increasing degree terms play a minor role near zero}}$, however, if we $\boxed{\text{include the second degree term}}$ we can see the parabolic nature of $\cosh x$. The parabola that best fits $\cosh x$ near zero is

$$\boxed{y = 1 + \frac{x^2}{2!}}.$$

Here's the partial graph.

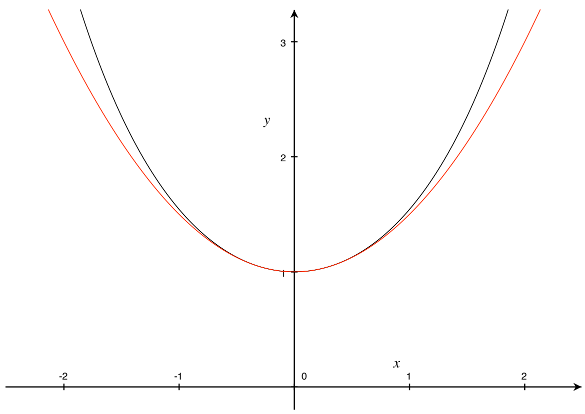

Figure 1: Partial graph of both $y = \cosh x$ (in black) and $y = 1 + x^2/2$ (in red).

6. $\boxed{\text{10 points}}$ By looking at the Taylor series, decide which of the folowing functions is largest, and which is smallest, for small positive θ.

$$1 + \sin\theta, \qquad \cos\theta, \qquad \frac{1}{1-\theta^2}$$

Solution: Using what we already know. For example, using what you learned in pre-calculus, especially about non-linear inequalities, might be insightful when looking at these expansions.

$$\begin{aligned}
1 + \sin\theta &= 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \cdots \\
\frac{1}{1-\theta^2} &= 1 + \theta^2 + \theta^4 + \theta^6 + \theta^{10} + \cdots
\end{aligned}$$

Clearly $\boxed{1 + \sin\theta \text{ is the largest}}$ amongst the three series when θ is positive and near zero. Now looking at the remaining two series it is clear that $\boxed{\cos\theta \text{ is smallest}}$. Here's a partial graph with the viewing window deliberately set to $x \in [-0.0078,\ 0.3047]$ and $y \in [0.8945,\ 1.1117]$. Again, please make every effort to learn how to use technology to help visualize complicated graphs.

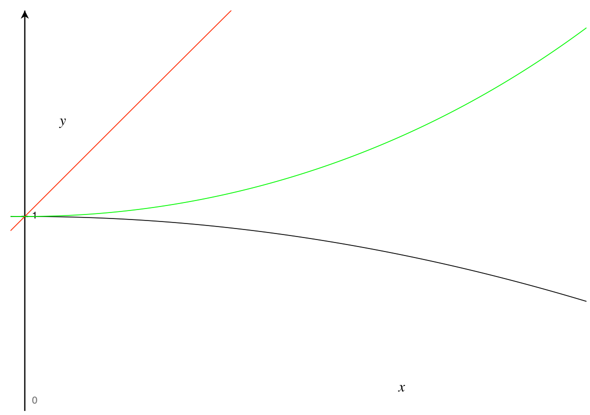

Figure 2: $y = 1 + \sin x$ in red; $y = \cos x$ in black; $y = (1 - x^2)^{-1}$ in green.

7. $\boxed{\text{10 points}}$ Find the sum of

$$\sum_{n=1}^{\infty} n x^{n-1}$$

for $|x| < 1$.

Solution: Okay, this may be a tough one, especially if you were taking partial sums and looking for patterns. However, this looks like it might be related to the following Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

To see this relationship, take the derivative.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 + x + x^2 + x^3 + x^4 + \cdots\right] \\ \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ \boxed{\frac{1}{(1-x)^2}} &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$