Name: ______________________________

Signature: ______________________________

Show all work clearly and in order, and box your final answers. Justify your answers whenever possible. You have 80 minutes to take this 100 point exam.

1. 10 points Use a *known* series to evaluate

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}.$$

Solution:

$$\begin{aligned}\frac{\sin x - x}{x^3} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - x}{x^3}\\ &= -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots\\ \lim_{x\to 0}\frac{\sin x - x}{x^3} &= \lim_{x\to 0}\left[-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots\right] = \boxed{-\frac{1}{6}}\end{aligned}$$

2. The **generalized Binomial Theorem** was discovered by Isaac Newton around 1665, and you probably learned the **Binomial Theorem** in pre-calculus. It was probably introduced as an expansion of $(a+b)^n$.

$$(a+b)^n = a^n + k_1a^{n-1}b + k_2a^{n-2}b^2 + k_3a^{n-3}b^3 + \cdots + k_{n-2}a^2b^{n-2} + k_{n-1}ab^{n-1} + b^n.$$

The pattern should be easy to follow, but the constants k_i are in fact difficult to compute. After some thought though, most students can figure out that these coefficients form a pattern for $n \in \mathbb{Z}^+$. As you may recall, the coefficients of the binomial expansion can be computed using this formula:

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!},$$

where n represents the degree (row in Pascal's Triangle) and r represents the position starting with 0 and ending with n in each expansion.

The **generalized Binomial Theorem** does not limit the power to being an integer though. We will be using the Taylor series to develop the **generalized Binomial Theorem** which states for any exponent $a \in \mathbb{R}$, integer $n \geq 0$, and $|x| < 1$:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots$$

Here we need to define the binomial coefficient

$$\binom{a}{n} = \frac{a\,(a-1)\,(a-2)\cdots(a-n+1)}{n!}.$$

For example

$$\binom{4/3}{3} = \frac{4/3\cdot(4/3-1)\,(4/3-2)}{3!} = -\frac{4}{81}.$$

So now, with what you know about Taylor series, try to develop the **generalized Binomial Theorem**.

(a) 10 points Repeatedly take derivatives of

$$f(x) = (1+x)^a,$$

and try to find a simple pattern for the n^{th} derivative.

Solution:

$$\begin{aligned}
f(x) &= (1+x)^a \\
f'(x) &= a\,(1+x)^{a-1} \\
f''(x) &= a\,(a-1)\,(1+x)^{a-2} \\
f'''(x) &= a\,(a-1)\,(a-2)\,(1+x)^{a-3} \\
\vdots &= \vdots \\
f^{(n)}(x) &= \boxed{a\,(a-1)\,(a-2)\cdots(a-n+1)\,(1+x)^{a-n}}
\end{aligned}$$

(b) 10 points Now we need to evaluate these derivatives at $x = 0$, and derive the **Taylor series** using this information.

Solution:

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= a \\
f''(0) &= a\,(a-1) \\
f'''(0) &= a\,(a-1)\,(a-2) \\
\vdots &= \vdots \\
f^{(n)}(0) &= a\,(a-1)\,(a-2)\cdots(a-n+1)
\end{aligned}$$

So the general coefficient of the **Taylor series** is

$$\frac{f^{(n)}(0)}{n!} = \frac{a\,(a-1)\,(a-2)\cdots(a-n+1)}{n!}$$

The result being:

$$\begin{aligned}(1+x)^a &= \boxed{1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n} x^n + \cdots} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n\end{aligned}$$

3. Given

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, \mathrm{d}x,$$

and the following graph.

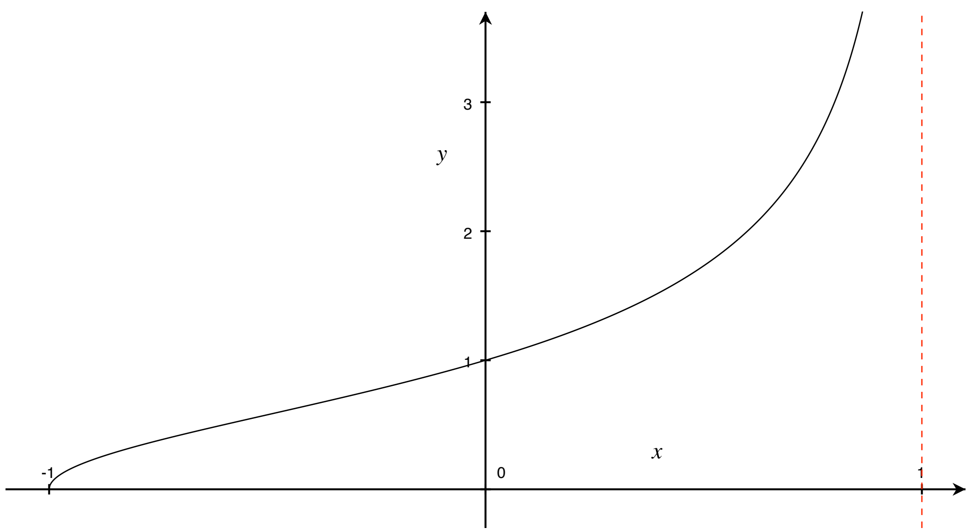

Figure 1: Partial graph of $y = \sqrt{\dfrac{1+x}{1-x}}$ and $x = 1$.

(a) $\boxed{\text{5 points}}$ Why is this an improper integral?

Solution: $\boxed{\text{At } x = 1 \text{ it is undefined.}}$

(b) $\boxed{\text{10 points}}$ Show that

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}},$$

if $-1 < x < 1$.

Solution:

$$\sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{1+x}{\sqrt{1-x^2}}$$

Or, if you prefer.

$$\frac{1+x}{\sqrt{1-x^2}} = \frac{1+x}{\sqrt{(1+x)(1-x)}} = \sqrt{\frac{1+x}{1-x}}$$

(c) 10 points Use

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}},$$

to evaluate

$$\int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x.$$

Solution:

$$\begin{aligned}\int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x &= \int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \boxed{\arcsin x - \sqrt{1-x^2} + C}\end{aligned}$$

(d) 10 points Evaluate

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x.$$

Solution:

$$\begin{aligned}\lim_{a\to 1^-}\int_{-1}^{a} \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x &= \lim_{a\to 1^-}\left(\arcsin x - \sqrt{1-x^2}\right)\Big]_{-1}^{a} \\ &= \lim_{a\to 1^-}\left[\arcsin a - \sqrt{1-a^2}\right] - \arcsin(-1) \\ &= \arcsin 1 - \arcsin(-1) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}\end{aligned}$$

4. (a) 10 points Use differentiation of a *known* power series to find the power series representation for

$$f(x) = \frac{1}{(1+x)^2}.$$

What is the radius of convergence?

Solution: You should note that we'll be using

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

to start this problem off. A bit *tricky*, but here goes ....

$$\begin{aligned} f(x) = \frac{1}{(1+x)^2} &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-(-x)}\right] \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[1 - x + x^2 - x^3 + \cdots\right] \\ &= 1 - 2x + 3x^2 - 4x^3 + \cdots \\ &= \boxed{\sum_{n=0}^{\infty}(-1)^n (n+1) x^n} \end{aligned}$$

You can use the **Ratio Test** to show that the $\boxed{\text{radius of convergence is } R = 1}$. By the way, the interval of convergence is $(-1,\ 1)$.

(b) $\boxed{\text{10 points}}$ Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}.$$

What is the radius of convergence?

Solution:

$$\begin{aligned} f(x) = \frac{1}{(1+x)^3} &= -\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(1+x)^2}\right] \\ &= -\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}(-1)^n (n+1) x^n\right] \\ &= -\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[1 - 2x + 3x^2 - 4x^3 + \cdots\right] \\ &= 1 - 3x + 6x^2 - 10x^3 + \cdots \\ &= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n (n+1)(n+2)}{2}x^n} \end{aligned}$$

You can use the **Ratio Test** to show that the $\boxed{\text{radius of convergence is } R = 1}$. By the way, the interval of convergence is $(-1,\ 1)$.

(c) 10 points Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}.$$

What is the radius of convergence?

Solution:

$$\begin{aligned} f(x) = \frac{x^2}{(1+x)^3} &= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n \\ &= x^2 \left[1 - 3x + 6x^2 - 10x^3 + \cdots\right] \\ &= \left[x^2 - 3x^3 + 6x^4 - 10x^5 + \cdots\right] \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^{n+2}} \end{aligned}$$

You can use the **Ratio Test** to show that the radius of convergence is $R = 1$. By the way, the interval of convergence is $(-1, 1)$.

5. Given that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

is a solution to the differential equation

$$f''(x) + f(x) = 0.$$

Answer the following questions.

(a) 10 points Find $f'(x)$.

Solution:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ f'(x) &= 0 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\ f'(x) &= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}} \end{aligned}$$

(b) 10 points Find $f''(x)$.

Solution:

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \\
f'(x) &= -\frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\
f''(x) &= -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
f''(x) &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}}
\end{aligned}
$$

(c) 10 points Expand both f and f'' and then substitute into the differential equation

$$f''(x) + f(x) = 0,$$

to verify that it is a solution.

Solution:

$$
\begin{aligned}
0 &= f''(x) + f(x) \\
0 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
0 &= \left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
0 &= 0
\end{aligned}
$$

6. 10 points Find the radius of convergence and the interval of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}.$$

Solution: Using the **Ratio Test**, we have

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!x^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)x}{(2n+1)}\right| \\
&= \frac{|x|}{2}
\end{aligned}
$$

So by the **Ratio Test**, the series converges for $|x| < 2$, so the $\boxed{\text{radius of convergence is } 2}$. To find the interval of convergence we need to test the endpoints ± 2. When $x = 2$ we get

$$\begin{aligned} a_n &= \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n \cdot 2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)} \\ &= \left(\frac{2}{1}\right) \cdot \left(\frac{4}{3}\right) \cdot \left(\frac{6}{5}\right) \cdot \left(\frac{8}{7}\right) \cdot \cdots \cdot \left(\frac{2n}{2n-1}\right) \end{aligned}$$

I think it is *obvious* that $a_n > 1$ for all n. Now if we were to use $x = -2$ we can still state that $|a_n| > 1$ for all n. So we have

$$\lim_{n \to \infty} |a_n| \neq 0,$$

and using the **Test for Divergence** we can state that the series diverges for $x = \pm 2$, so the $\boxed{\text{interval of convergence is } (-2,\ 2)}$.

7. $\boxed{\text{10 points}}$ Given that

$$f(x) = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}.$$

Try to find a power series for $f(x)$ and its interval of convergence.

Solution: You should note that we'll be using

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

and

$$\begin{aligned} \frac{2}{1-2x} = 2\left[\frac{1}{1-(2x)}\right] &= 2\left[1 + 2x + 4x^2 + 8x^3 + \cdots + (2x)^n + \cdots\right] \\ &= 2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots \end{aligned}$$

so

$$\begin{aligned} \frac{2}{1-2x} - \frac{1}{1-x} &= \left[2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots\right] - \left[1 + x + x^2 + \cdots\right] \\ &= 1 + 3x + 7x^2 + 15x^3 + \cdots + \left(2^{n+1} - 1\right)x^n + \cdots \\ &= \sum_{n=0}^{\infty} \left(2^{n+1} - 1\right)x^n \end{aligned}$$

Using the **Ratio Test** the interval of convergence is $(-1/2,\ 1/2)$.

8. 10 points **Euler's identity** is helpful when dealing with complex numbers, it states that for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

You should know from basic algebra how to raise i (where $\sqrt{-1} = i$) to a natural number power. Here's a short list:

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i, \ldots.$$

Let's use our series expansion for e^x, but this time let's replace x by $i\theta$ and see if you can get **Euler's identity** by doing this. Use this identity to find out a simple form (it's a very simple and important number) for $e^{i\pi}$.

Solution:

$$\begin{aligned}
e^{\theta} &= 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \cdots \\
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - i\frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

Using **Euler's identity** with $\theta = \pi$ we get:

$$\begin{aligned}
e^{i\pi} &= \cos\pi + i\sin\pi \\
e^{i\pi} &= -1 + i\cdot 0 \\
e^{i\pi} &= -1
\end{aligned}$$

This, of course leads to a very important relationship between π, 1, 0, e and i, namely

$$\boxed{e^{i\pi} + 1 = 0}.$$

These five numbers are often referred to as the five most important numbers in mathematics. [Hate to sound biased, but these numbers are really in a class by themselves. In a way, numbers are polytheistic, and these five numbers stand equally above all others.] When I travel I almost always visit cultural centers where people are asked to leave a comment in a guest book, and I often leave $e^{i\pi} + 1 = 0$ because those symbols are universally understood by educated people. Once I saw it crossed off (with a comment stating that it was not true), I guess because it offended someone. Anyway, I just might start leaving $1 = 0.\overline{999}$ which probably appears equally offensive—I know it once offended me when I first saw calculus.