Name: ______________________________

Signature: ______________________________

Show all work clearly and in order, and box your final answers. Justify your answers whenever possible. You have 80 minutes to take this 100 point exam.

1. 10 points Use a *known* series to evaluate

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}.$$

2. The **generalized Binomial Theorem** was discovered by Isaac Newton around 1665, and you probably learned the **Binomial Theorem** in pre-calculus. It was probably introduced as an expansion of $(a+b)^n$.

$$(a+b)^n = a^n + k_1 a^{n-1}b + k_2 a^{n-2}b^2 + k_3 a^{n-3}b^3 + \cdots + k_{n-2}a^2b^{n-2} + k_{n-1}ab^{n-1} + b^n.$$

The pattern should be easy to follow, but the constants k_i are in fact difficult to compute. After some thought though, most students can figure out that these coefficients form a pattern for $n \in \mathbb{Z}^+$. As you may recall, the coefficients of the binomial expansion can be computed using this formula:

$${}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!},$$

where n represents the degree (row in Pascal's Triangle) and r represents the position starting with 0 and ending with n in each expansion.

The **generalized Binomial Theorem** does not limit the power to being an integer though. We will be using the Taylor series to develop the **generalized Binomial Theorem** which states for any exponent $a \in \mathbb{R}$, integer $n \geq 0$, and $|x| < 1$:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots$$

Here we need to define the binomial coefficient

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}.$$

For example

$$\binom{4/3}{3} = \frac{4/3\cdot(4/3-1)(4/3-2)}{3!} = -\frac{4}{81}.$$

So now, with what you know about Taylor series, try to develop the **generalized Binomial Theorem**.

(a) 10 points Repeatedly take derivatives of

$$f(x) = (1+x)^a,$$

and try to find a simple pattern for the n^{th} derivative.

(b) 10 points Now we need to evaluate these derivatives at $x = 0$, and derive the **Taylor series** using this information.

3. Given

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, \mathrm{d}x,$$

and the following graph.

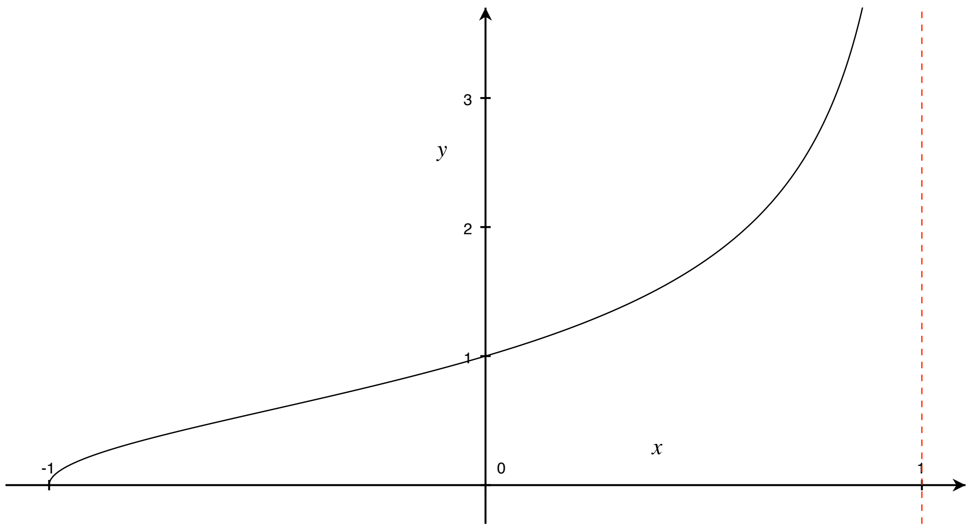

Figure 1: Partial graph of $y = \sqrt{\dfrac{1+x}{1-x}}$ and $x = 1$.

(a) 5 points Why is this an improper integral?

(b) 10 points Show that

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}},$$

if $-1 < x < 1$.

(c) 10 points Use

$$\sqrt{\frac{1+x}{1-x}} = \frac{1+x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}},$$

to evaluate

$$\int \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x.$$

(d) 10 points Evaluate

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\ \mathrm{d}x.$$

4. (a) 10 points Use differentiation of a *known* power series to find the power series representation for

$$f(x) = \frac{1}{(1+x)^2}.$$

What is the radius of convergence?

(b) 10 points Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}.$$

What is the radius of convergence?

(c) 10 points Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}.$$

What is the radius of convergence?

5. Given that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

is a solution to the differential equation

$$f''(x) + f(x) = 0.$$

Answer the following questions.

(a) 10 points Find $f'(x)$.

(b) 10 points Find $f''(x)$.

(c) 10 points Expand both f and f'' and then substitute into the differential equation

$$f''(x) + f(x) = 0,$$

to verify that it is a solution.

6. 10 points Find the radius of convergence and the interval of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}.$$

7. 10 points Given that

$$f(x) = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}.$$

Try to find a power series for $f(x)$ and its interval of convergence.

8. 10 points **Euler's identity** is helpful when dealing with complex numbers, it states that for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

You should know from basic algebra how to raise i (where $\sqrt{-1} = i$) to a natural number power. Here's a short list:

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i, \ldots.$$

Let's use our series expansion for e^x, but this time let's replace x by $i\theta$ and see if you can get **Euler's identity** by doing this. Use this identity to find out a simple form (it's a very simple and important number) for $e^{i\pi}$.