1

FUNCTIONS AND MODELS

1.2
MATHEMATICAL MODELS: A CATALOG OF ESSENTIAL FUNCTIONS

In this section, we will learn about:
The purpose of mathematical models.

MATHEMATICAL MODELS

A mathematical model is a mathematical description—often by means of a function or an equation—of a real-world phenomenon such as:

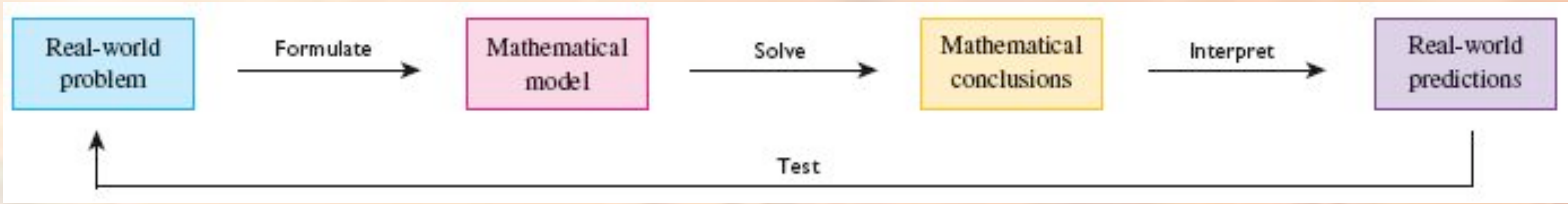
- ♣ Size of a population
- ♣ Demand for a product
- ♣ Speed of a falling object
- ♣ Life expectancy of a person at birth
- ♣ Cost of emission reductions

PURPOSE

The purpose of the model is to understand the phenomenon and, perhaps, to make predictions about future behavior.

PROCESS

The figure illustrates the process of mathematical modeling.



STAGE 1

Given a real-world problem, our first task is to formulate a mathematical model.

- We do this by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable.

STAGE 1

We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables.

- In situations where there is no physical law to guide us, we may need to collect data—from a library, the Internet, or by conducting our own experiments—and examine the data in the form of a table in order to discern patterns.

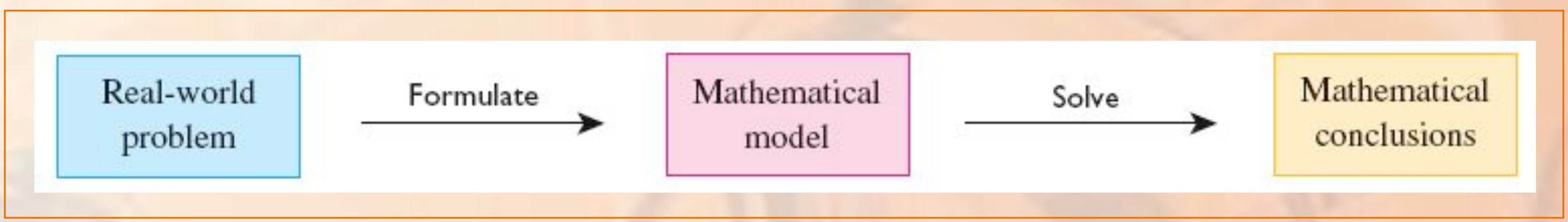
STAGE 1

From this numerical representation of a function, we may wish to obtain a graphical representation by plotting the data.

- In some cases, the graph might even suggest a suitable algebraic formula.

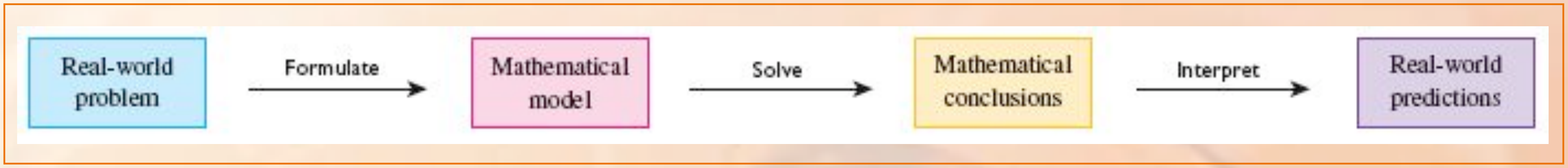
STAGE 2

The second stage is to apply the mathematics that we know—such as the calculus that will be developed throughout this book—to the model that we have formulated in order to derive mathematical conclusions.



STAGE 3

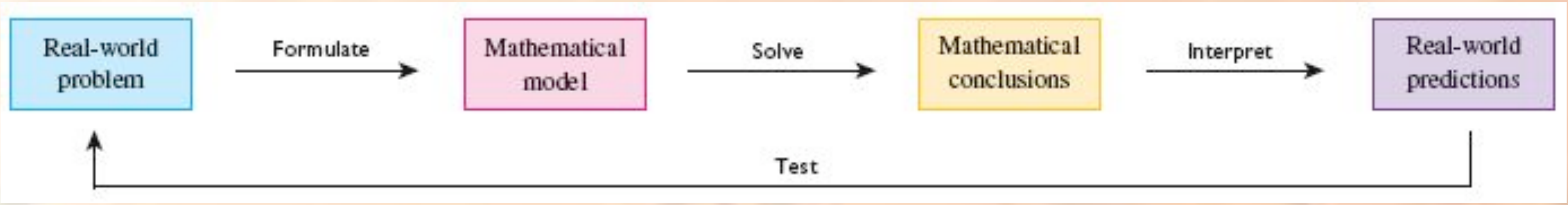
In the third stage, we take those conclusions and interpret them as information about the original real-world phenomenon—by way of offering explanations or making predictions.



STAGE 4

The final step is to test our predictions by checking against new real data.

- If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.



MATHEMATICAL MODELS

A mathematical model is never a completely accurate representation of a physical situation—it is an idealization.

- ♣ A good model simplifies reality enough to permit mathematical calculations, but is accurate enough to provide valuable conclusions.
- ♣ It is important to realize the limitations of the model.
- ♣ In the end, Mother Nature has the final say.

MATHEMATICAL MODELS

There are many different types of functions that can be used to model relationships observed in the real world.

- In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

When we say that *y* is a linear function of *x*, we mean that the graph of the function is a line.

- So, we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

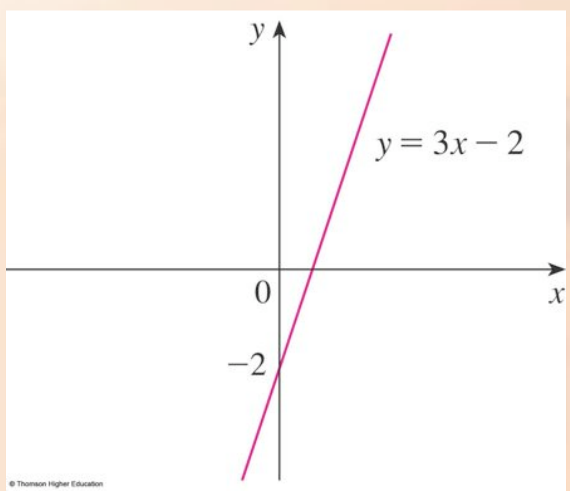
where *m* is the slope of the line and *b* is the *y*-intercept.

A characteristic feature of linear functions is that they grow at a constant rate.

LINEAR MODELS

For instance, the figure shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values.

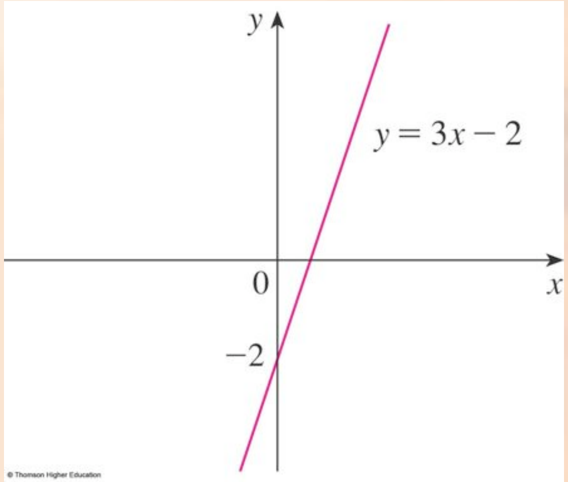
- Notice that, whenever x increases by 0.1, the value of $f(x)$ increases by 0.3.
- So, $f(x)$ increases three times as fast as x.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

LINEAR MODELS

- Thus, the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.



x	$f(x) = 3x - 2$
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1.3	1.9
1.4	2.2
1.5	2.5

As dry air moves upward, it expands and cools.

- If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature *T* (in °C) as a function of the height *h* (in kilometers), assuming that a linear model is appropriate.
- Draw the graph of the function in part (a). What does the slope represent?
- What is the temperature at a height of 2.5 km?

 Example 1 a

As we are assuming that T is a linear function of h, we can write $T = mh + b.$

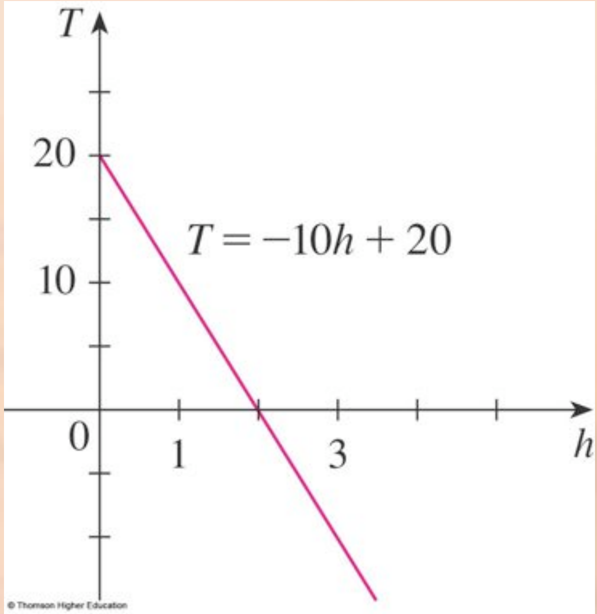
- We are given that $T = 20$ when $h = 0$, so $20 = m \cdot 0 + b = b.$
- In other words, the y-intercept is $b = 20$.
- We are also given that $T = 10$ when $h = 1$, so $10 = m \cdot 1 + 20$
- Thus, the slope of the line is m = 10 – 20 = -10.
- The required linear function is $T = -10h + 20$.

LINEAR MODELS

Example 1 b

The slope is m = -10˚C/km.

This represents the rate of change of temperature with respect to height.

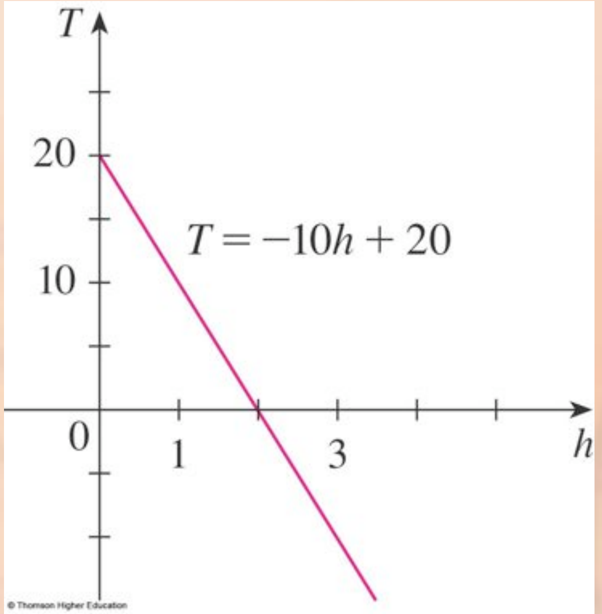


Example 1 c

At a height of h = 2.5 km,

the temperature is:

$T = -10(2.5) + 20 = -5^{\circ}C.$



EMPIRICAL MODEL

If there is no physical law or principle to help us formulate a model, we construct an empirical model.

- This is based entirely on collected data.
- We seek a curve that ‘fits’ the data in the sense that it captures the basic trend of the data points.

The table lists the average carbon dioxide (CO_2) level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2002.

Use the data to find a model for the CO_2 level.

TABLE 1

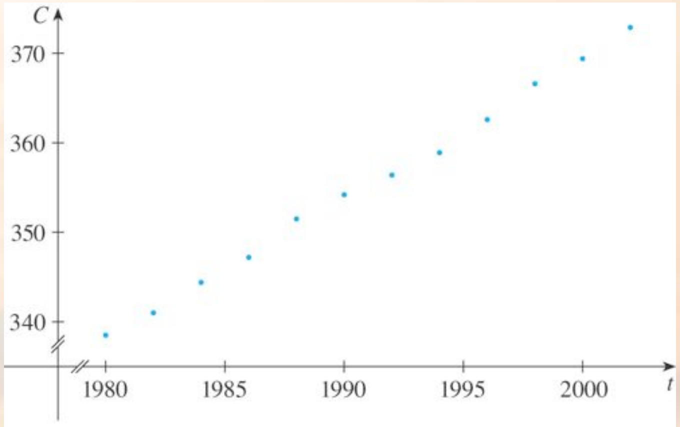
Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1992	356.4
1982	341.1	1994	358.9
1984	344.4	1996	362.6
1986	347.2	1998	366.6
1988	351.5	2000	369.4
1990	354.2	2002	372.9

LINEAR MODELS

Example 2

We use the data in the table to make the scatter plot shown in the figure.

- In the plot, *t* represents time (in years) and *C* represents the CO_2 level (in parts per million, ppm).



Scatter plot for the average CO_2 level

TABLE 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1992	356.4
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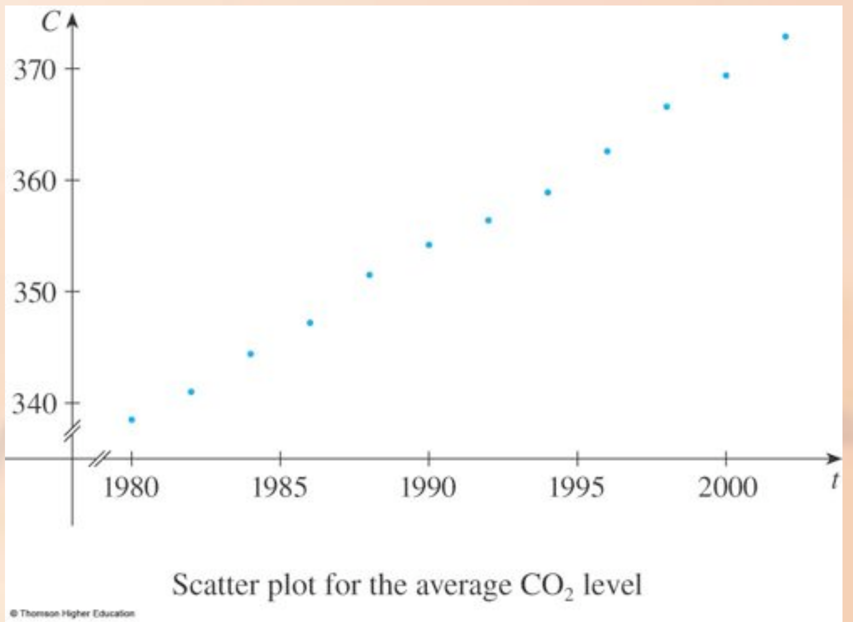
1990	354.2	2002	372.9

Notice that the data points appear to lie close to a straight line.

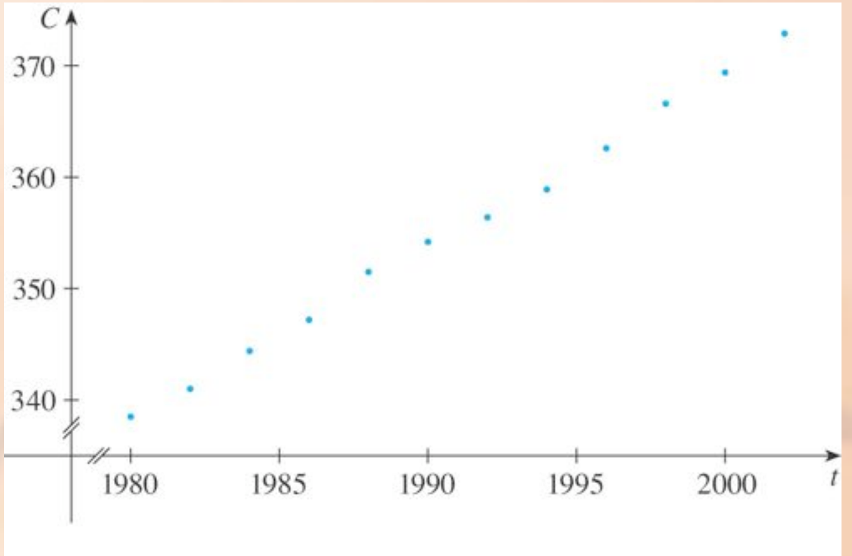
- So, in this case, it's natural to choose a linear model.



Scatter plot for the average CO_2 level

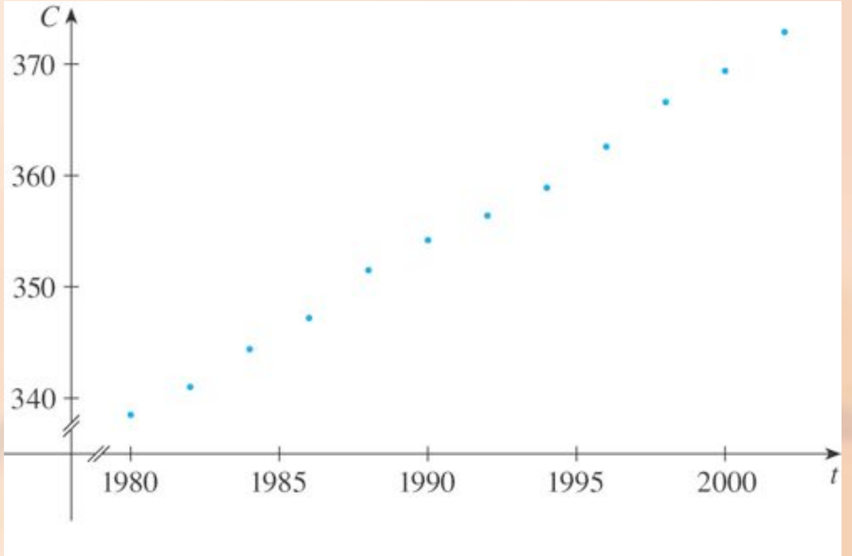
However, there are many possible lines that approximate these data points.

- ♣ So, which one should we use?



Scatter plot for the average CO_2 level

One possibility is the line that passes through the first and last data points.

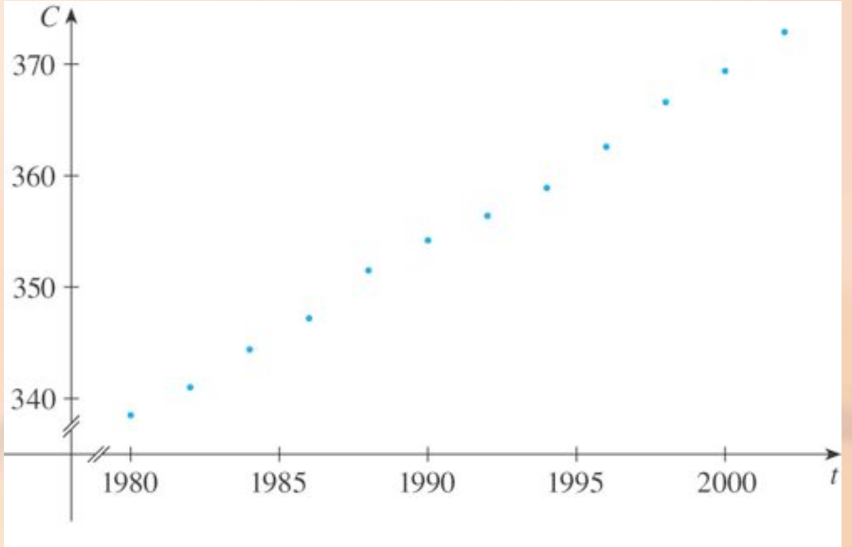


Scatter plot for the average CO_2 level

Example 2

The slope of this line is:

$$\frac{372.9-338.7}{2002-1980}=\frac{34.2}{22}\approx 1.5545$$

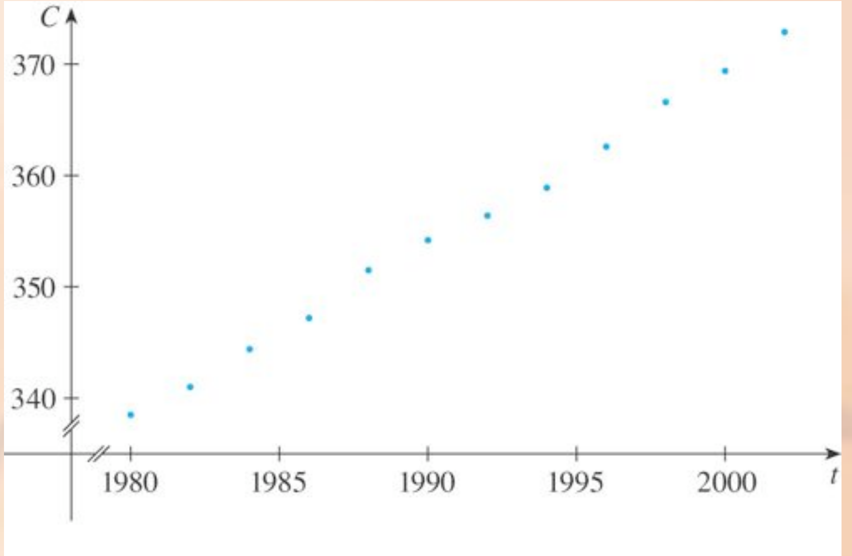


Scatter plot for the average CO_2 level

The equation of the line is:

$C - 338.7 = 1.5545(t - 1980)$

or $C = 1.5545t - 2739.21$



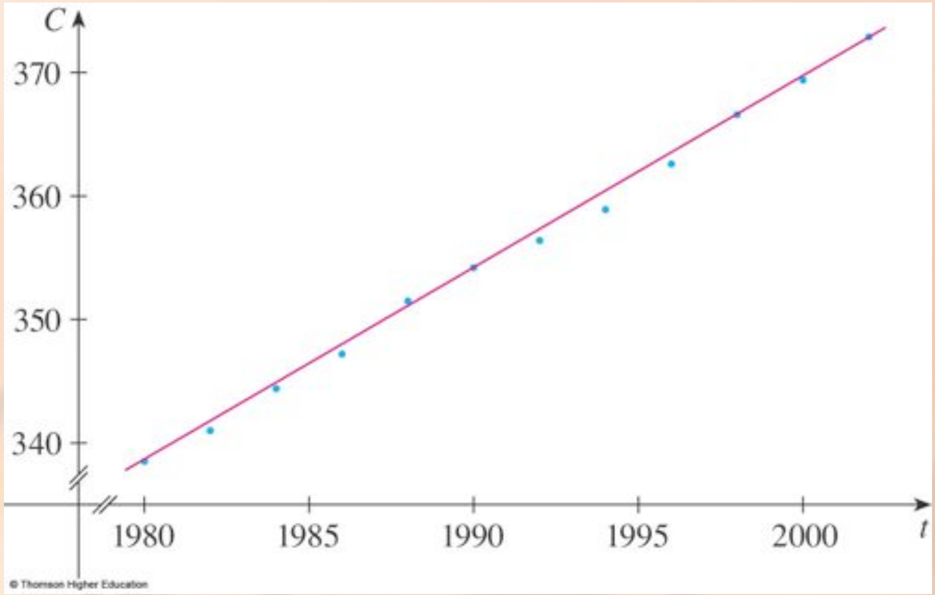
Scatter plot for the average CO_2 level

LINEAR MODELS

Example 2

This equation gives one possible linear model for the CO_2 level.

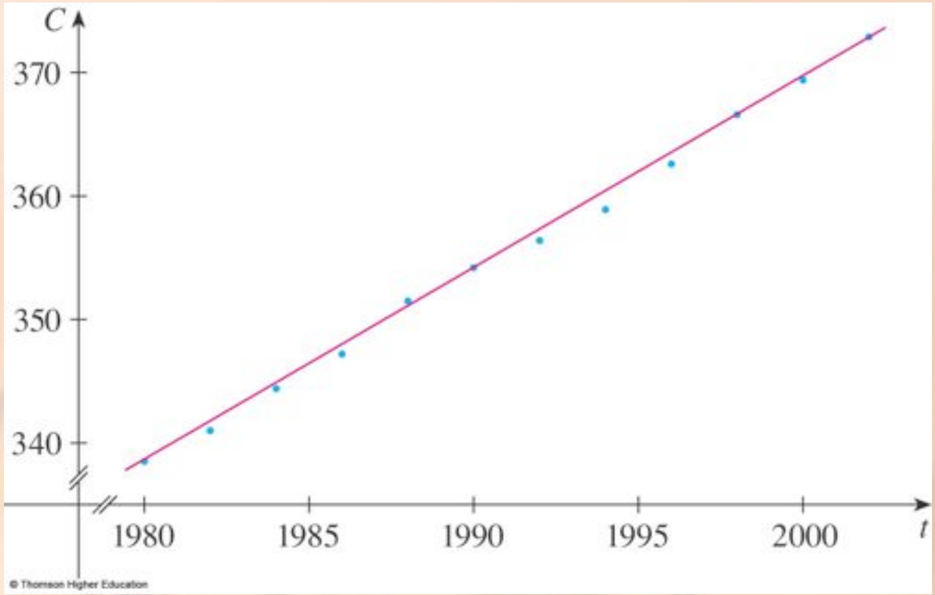
It is graphed in the figure.



LINEAR MODELS

Example 2

Although our model fits the data reasonably well, it gives values higher than most of the actual CO_2 levels.



A better linear model is obtained by a procedure from statistics called linear regression.

Example 2

If we use a graphing calculator, we enter the data from the table into the data editor and choose the linear regression command.

- With Maple, we use the fit[leastsquare] command in the stats package.
- With Mathematica, we use the Fit command.

TABLE 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1992	356.4
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E. g. 2—Equation 2

The machine gives the slope and *y*-intercept of the regression line as:

$m = 1.55192$ $b = -2734.55$

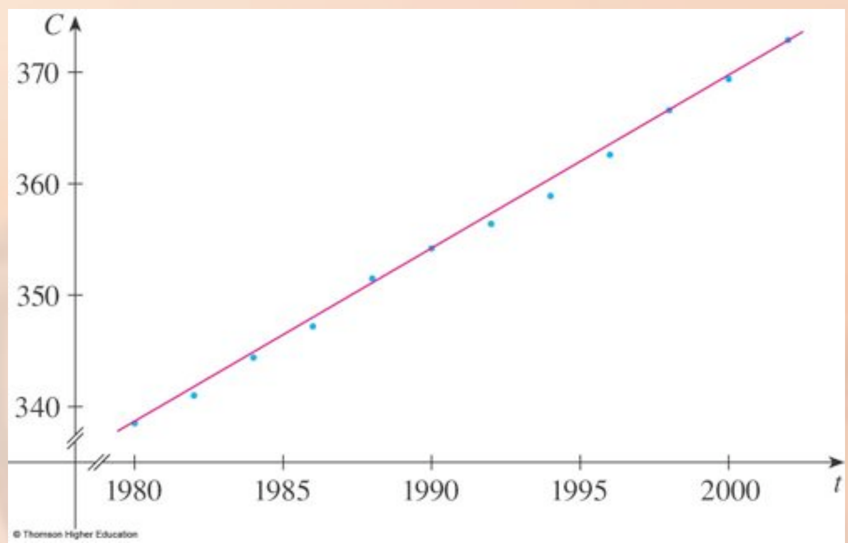
So, our least squares model for the level CO_2 is:

$$C = 1.55192t - 2734.55$$

LINEAR MODELS

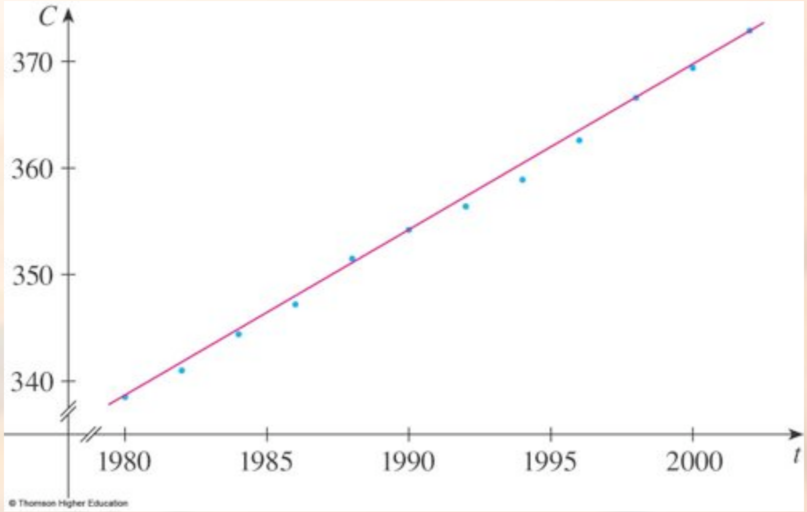
Example 2

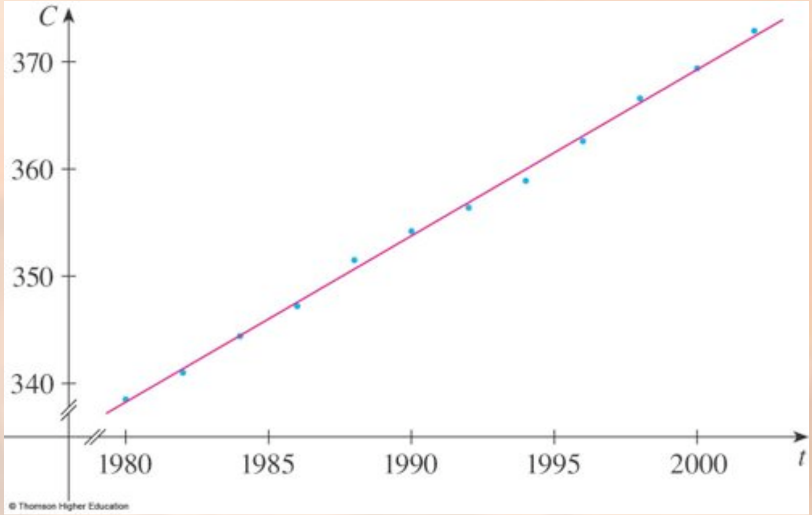
In the figure, we graph the regression line as well as the data points.



Example 2

Comparing with the earlier figure, we see that it gives a better fit than our previous linear model.





Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for 2010.

- According to this model, when will the CO_2 level exceed 400 parts per million?

Using Equation 2 with t = 1987, we estimate that the average CO_2 level in 1987 was:

$$C(1987) = (1.55192)(1987) - 2734.55 \approx 349.12$$

- This is an example of interpolation—as we have estimated a value between observed values.
- In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm.
- So, our estimate is quite accurate.

With $t = 2010$, we get:

$$C(2010) = (1.55192)(2010) - 2734.55 \approx 384.81$$

So, we predict that the average CO_2 level in 2010 will be 384.8 ppm.

- This is an example of extrapolation—as we have predicted a value outside the region of observations.
- Thus, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO_2 level exceeds 400 ppm when:

$$1.55192t - 2734.55 > 400$$

Solving this inequality, we get:

$$t > \frac{3134.55}{1.55192} \approx 2019.79$$

- Thus, we predict that the CO_2 level will exceed 400 ppm by 2019.
- This prediction is somewhat risky—as it involves a time quite remote from our observations.

POLYNOMIALS

A function *P* is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where *n* is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial.

POLYNOMIALS

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$.

If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is *n*.

- For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

DEGREE 1

A polynomial of degree 1 is of the form

$P(x) = mx + b$

So, it is a linear function.

DEGREE 2

A polynomial of degree 2 is of the form

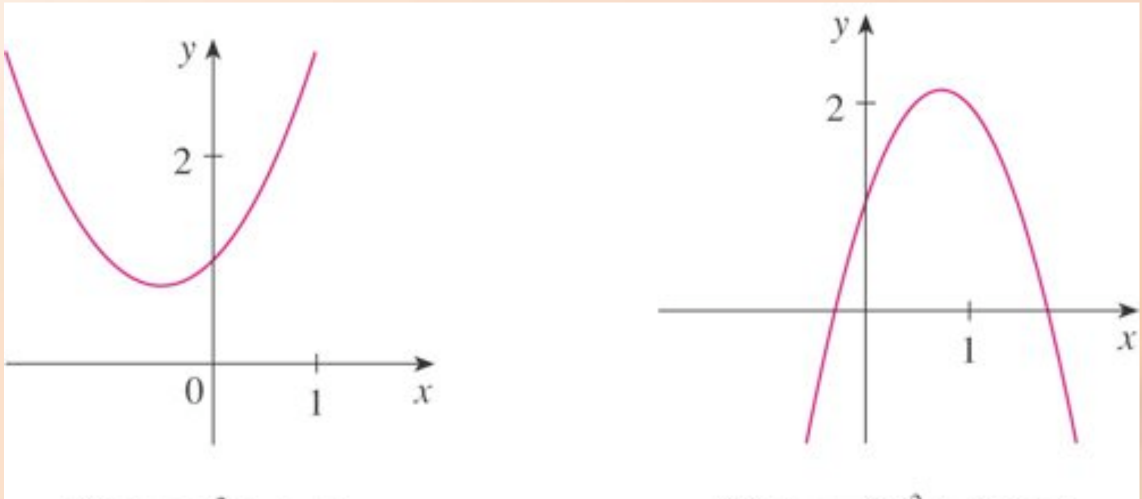
$P(x) = ax^2 + bx + c$

It is called a quadratic function.

DEGREE 2

Its graph is always a parabola obtained by shifting the parabola $y = x^2$.

- The parabola opens upward if $a > 0$ and downward if $a < 0$.



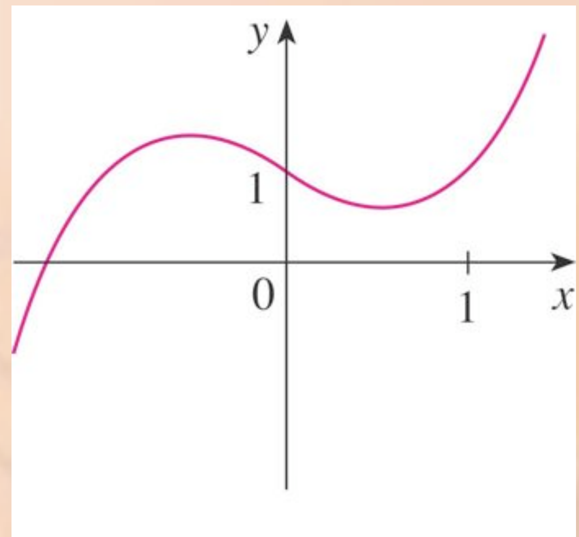
(a) $y = x^2 + x + 1$ (b) $y = -2x^2 + 3x + 1$

DEGREE 3

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad (a \neq 0)$$

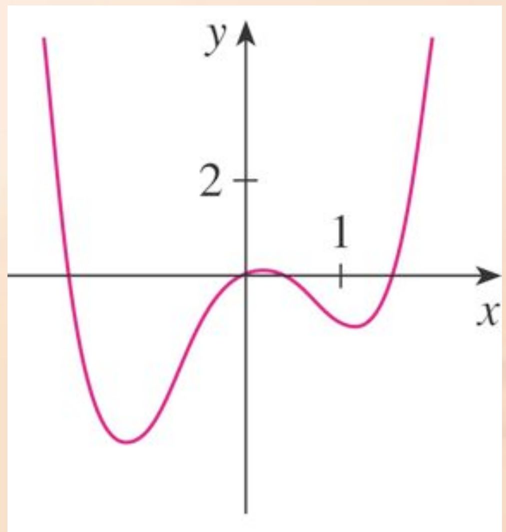
It is called a cubic function.



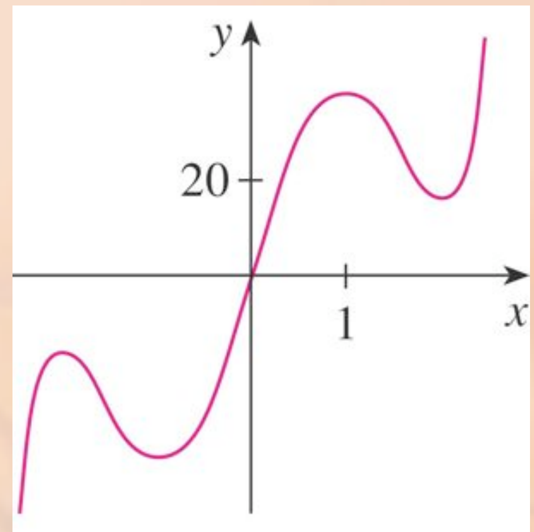
(a) $y = x^3 - x + 1$

DEGREES 4 AND 5

The figures show the graphs of polynomials of degrees 4 and 5.

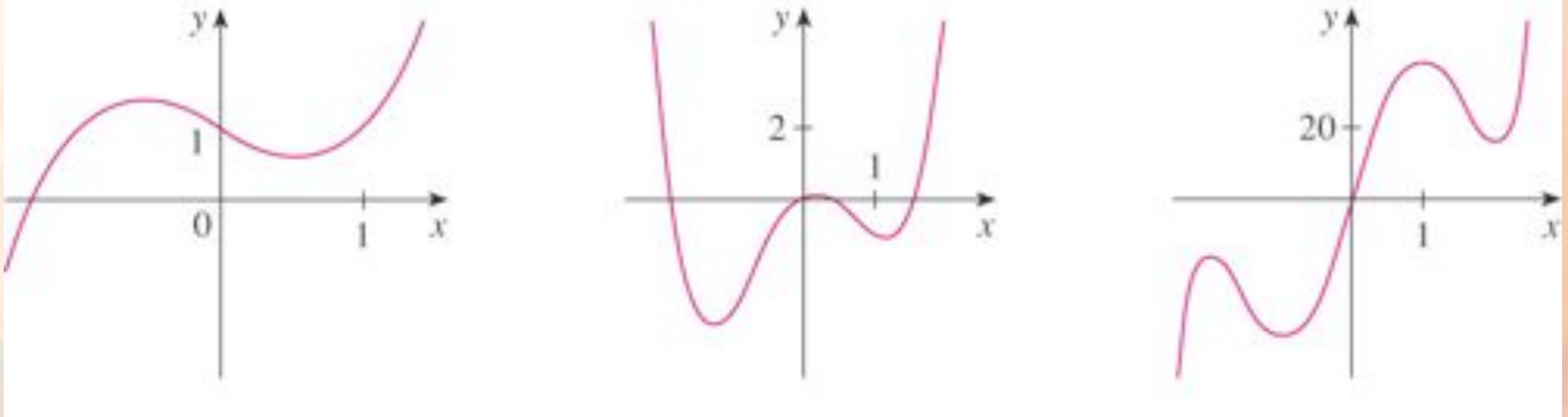


(b) $y = x^4 - 3x^2 + x$



(c) $y = 3x^5 - 25x^3 + 60x$

We will see later why these three graphs have these shapes.



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Polynomials are commonly used to model various quantities that occur in the natural and social sciences.

- For instance, in Section 3.7, we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity.
- In the following example, we use a quadratic function to model the fall of a ball.

Example 4

A ball is dropped from the upper observation deck of the CN Tower—450 m above the ground—and its height *h* above the ground is recorded at 1-second intervals.

- Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Example 4

We draw a scatter plot of the data.

We observe that a linear model is inappropriate.

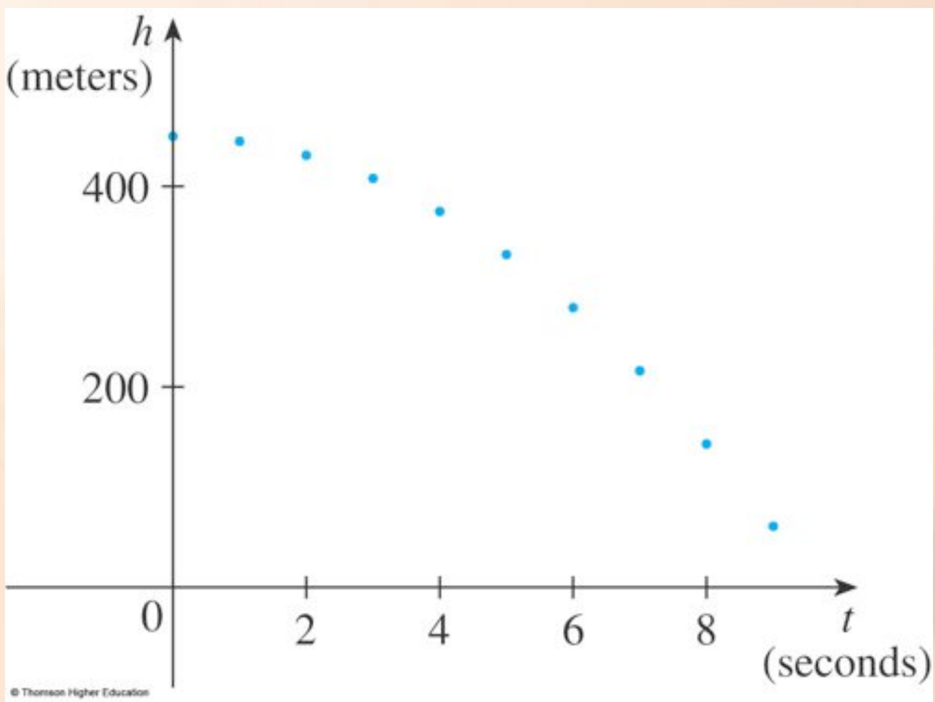
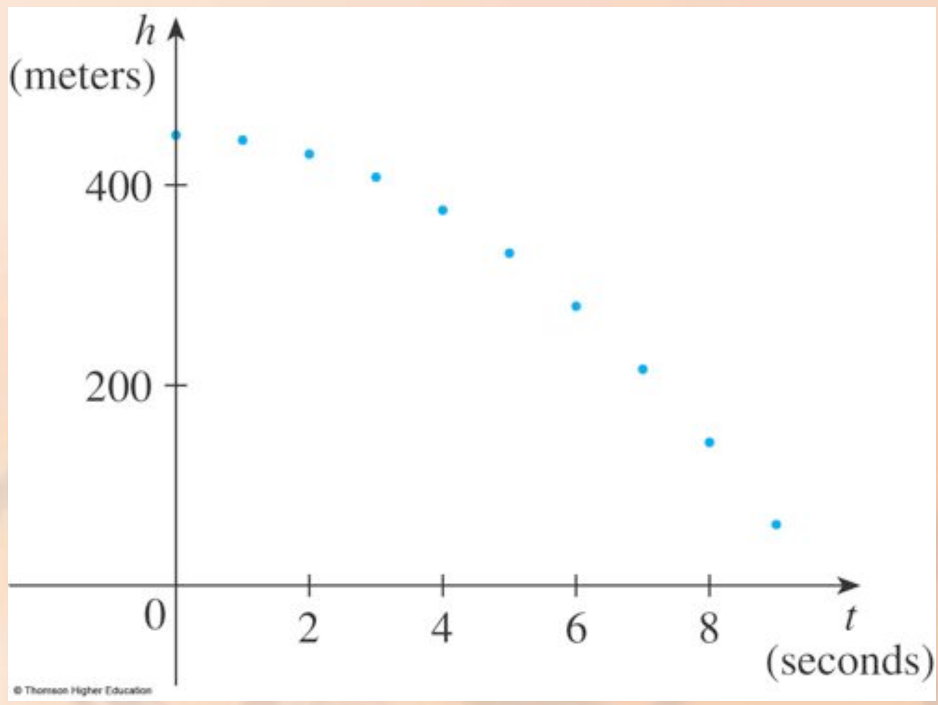


TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

However, it looks as if the data points might lie on a parabola.

- So, we try a quadratic model instead.

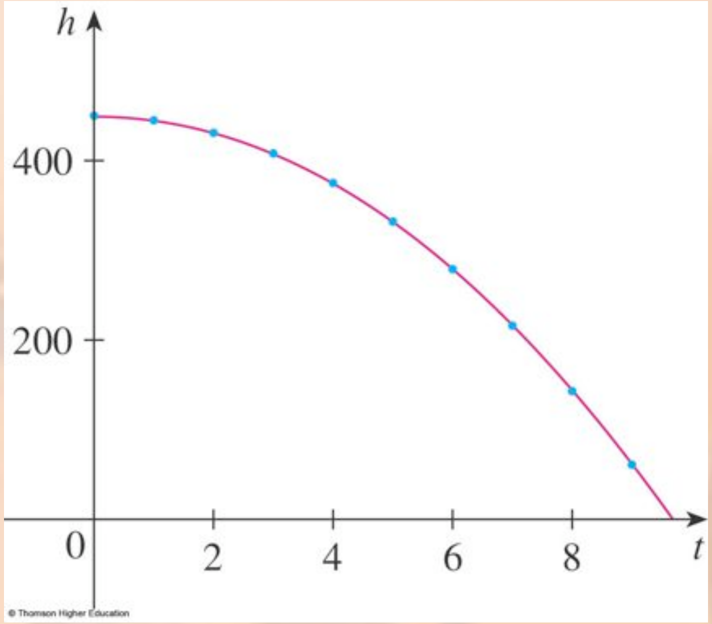


Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$h = 449.36 + 0.96t - 4.90t^2$$

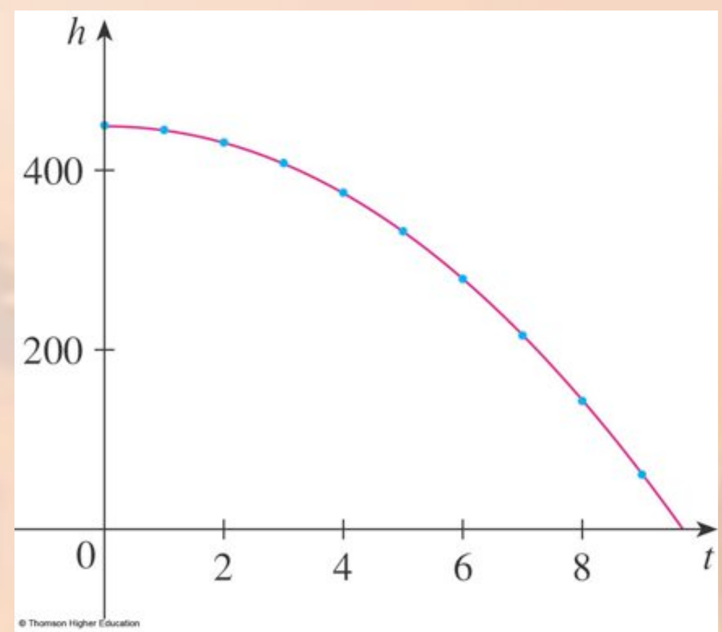
We plot the graph of Equation 3 together with the data points.

We see that the quadratic model gives a very good fit.



The ball hits the ground when $h = 0$.

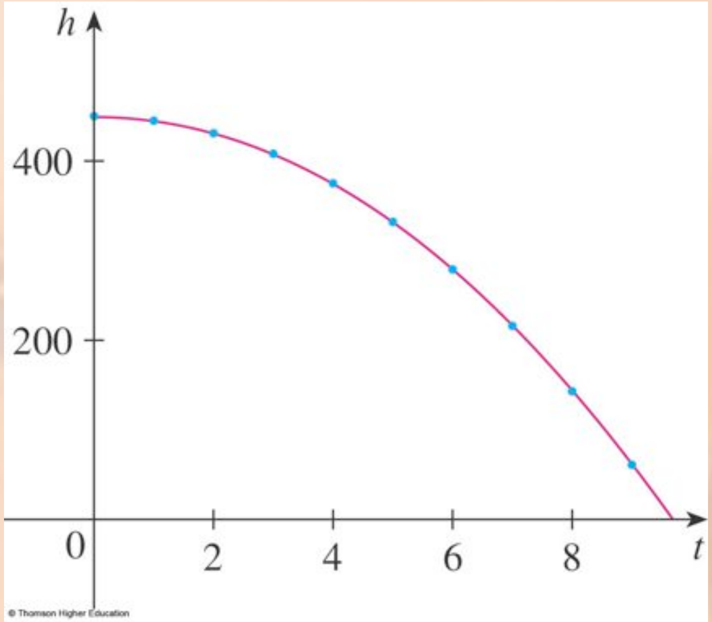
So, we solve the quadratic equation
$-4.90t^2 + 0.96t + 449.36 = 0$



The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

- The positive root is $t \approx 9.67$
- So, we predict the ball will hit the ground after about 9.7 seconds.



POWER FUNCTIONS

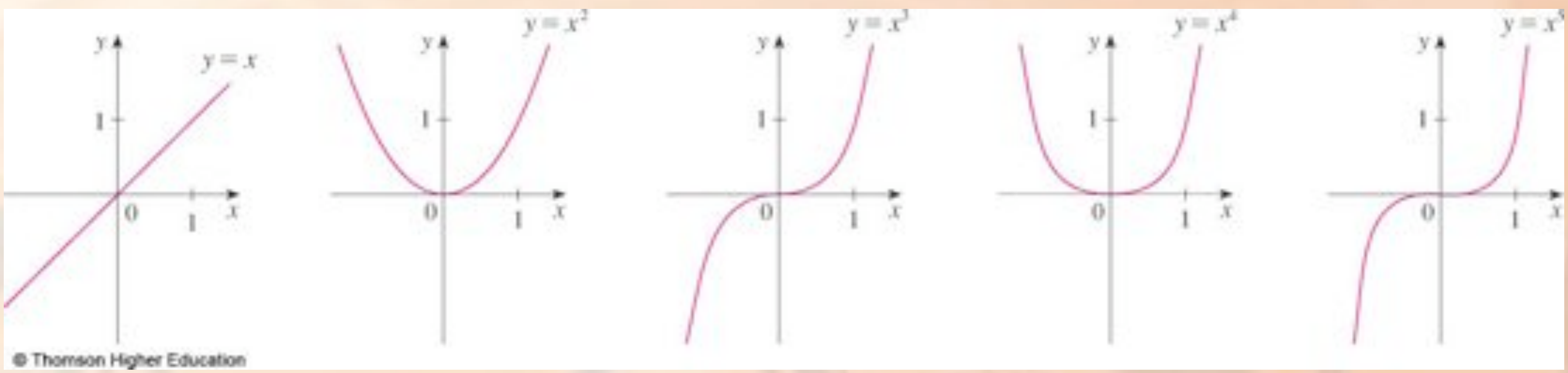
A function of the form $f(x) = x^a$, where a is constant, is called a power function.

- ♣ We consider several cases.

CASE 1

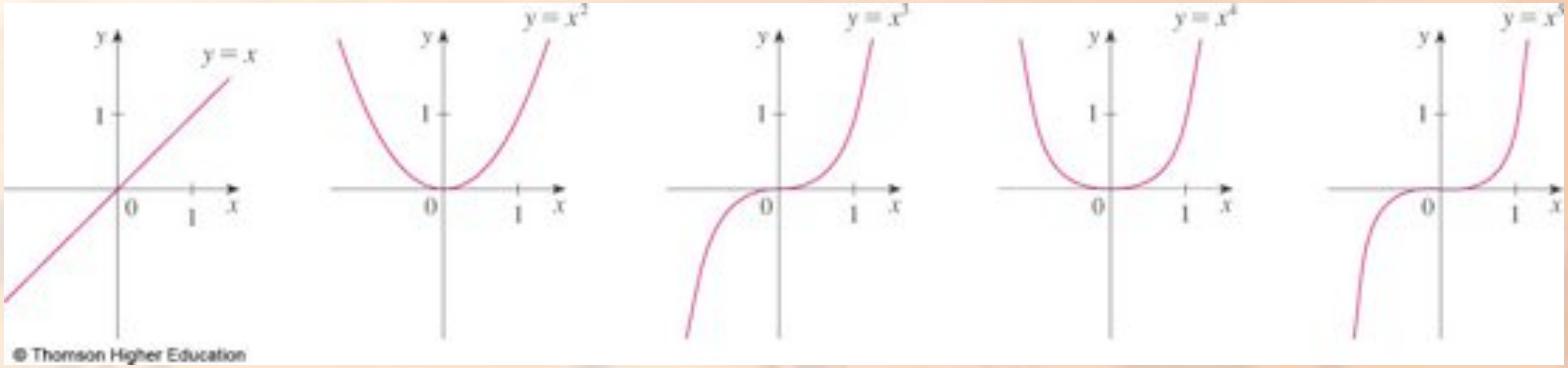
$a = n$, where n is a positive integer

- The graphs of $f(x) = x^n$ for n = 1, 2, 3, 4, and 5 are shown.
- These are polynomials with only one term.



CASE 1

- We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

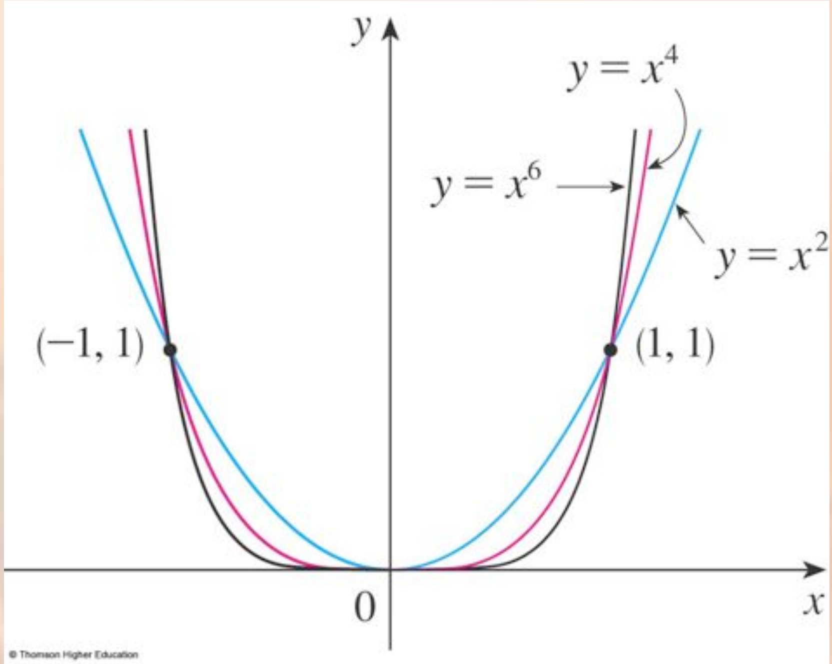


CASE 1

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

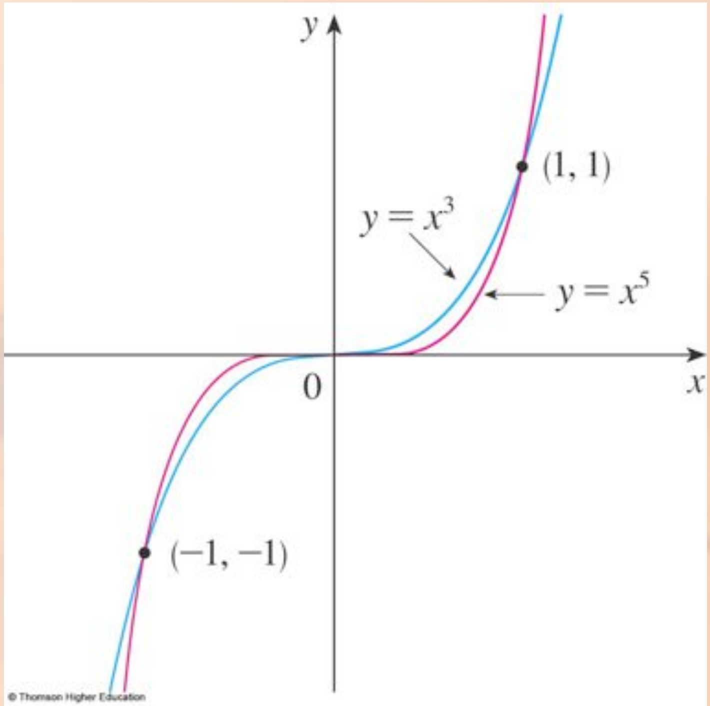
CASE 1

If n is even, then $f(x) = x^n$ is an even function, and its graph is similar to the parabola $y = x^2$.



CASE 1

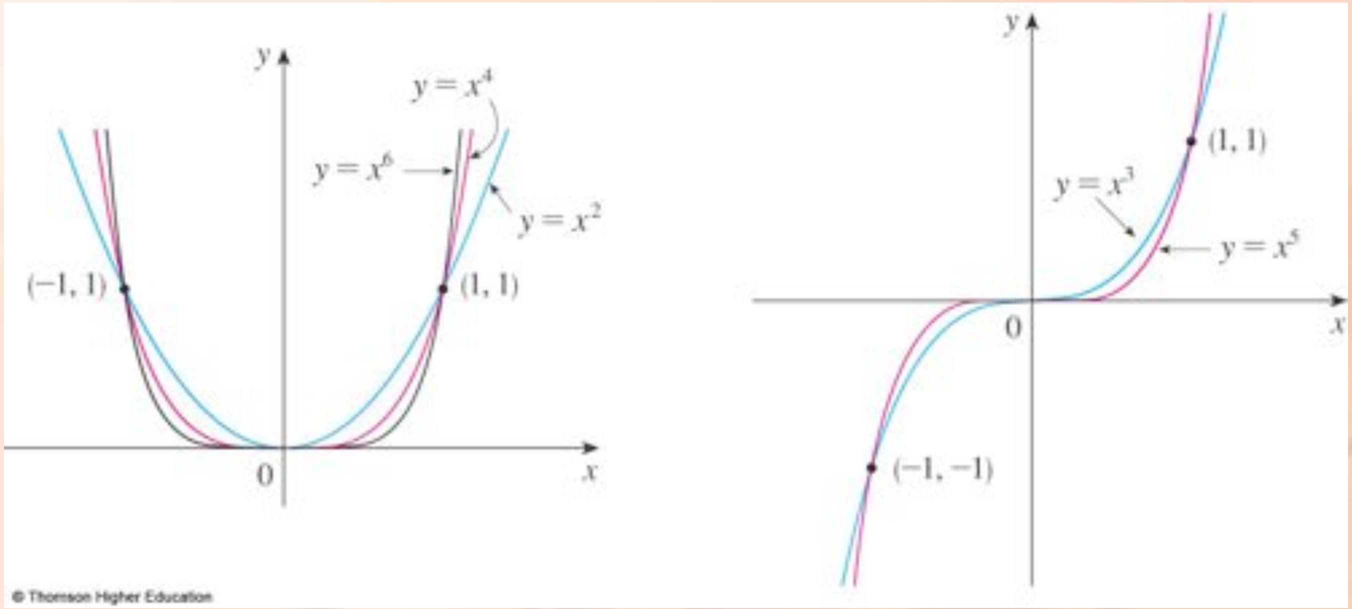
If n is odd, then $f(x) = x^n$ is an odd function, and its graph is similar to that of $y = x^3$.



CASE 1

However, notice from the figure that, as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$.

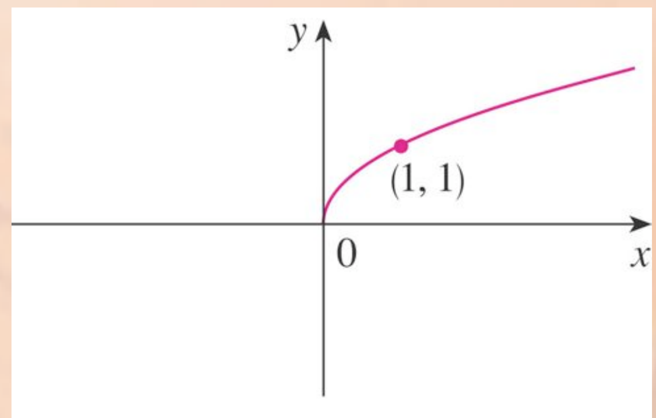
- If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.



CASE 2

$a = 1/n$, where n is a positive integer

- The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a root function.
- For $n = 2$, it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$.
- For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

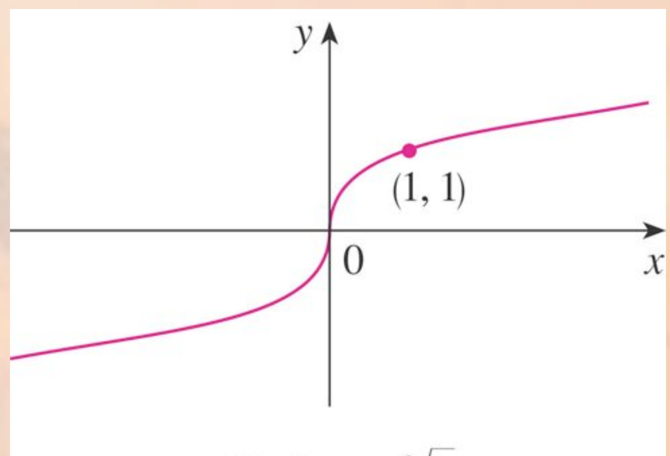


(a) $f(x) = \sqrt{x}$

CASE 2

For $n = 3$, we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown.

- The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.

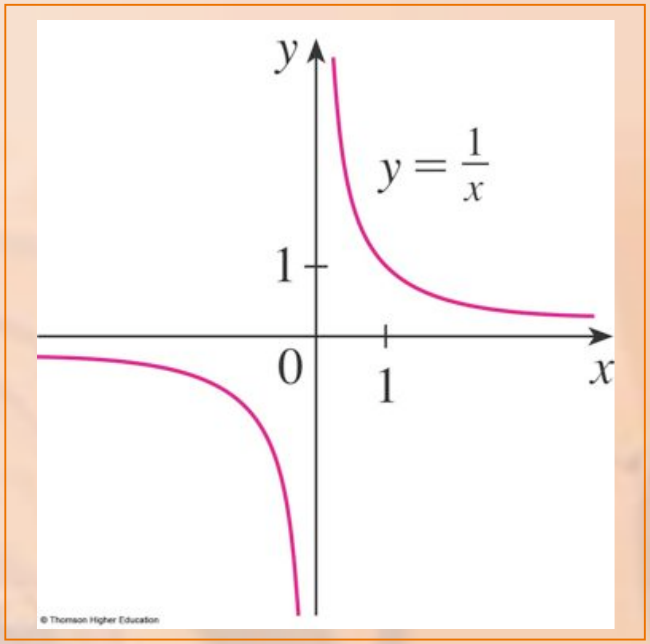


(b) $f(x) = \sqrt[3]{x}$

CASE 3

a = -1

- The graph of the reciprocal function $f(x) = x^{-1} = 1/x$ is shown.
- Its graph has the equation $y = 1/x$, or $xy = 1$.
- It is a hyperbola with the coordinate axes as its asymptotes.



CASE 3

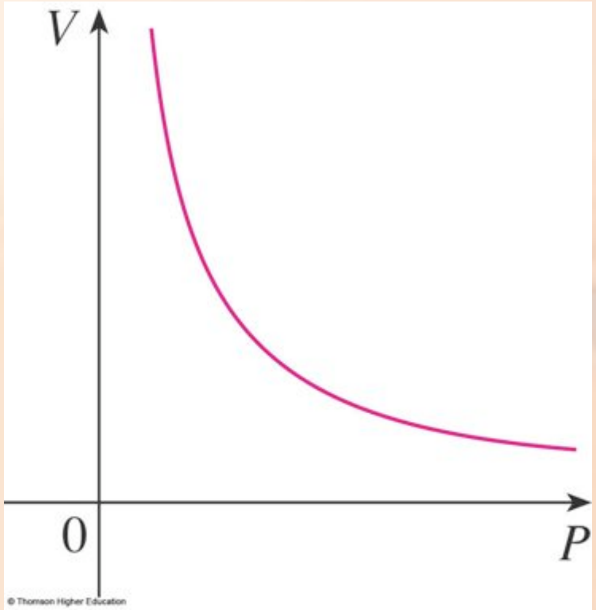
This function arises in physics and chemistry in connection with Boyle's Law, which states that, when the temperature is constant, the volume *V* of a gas is inversely proportional to the pressure *P*.

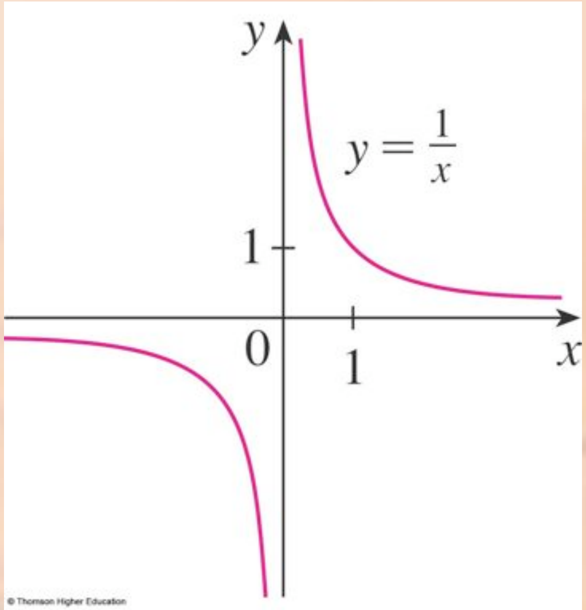
$$V = \frac{C}{P}$$

where *C* is a constant.

CASE 3

So, the graph of *V* as a function of *P* has the same general shape as the right half of the previous figure.





RATIONAL FUNCTIONS

A rational function *f* is a ratio of two polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$

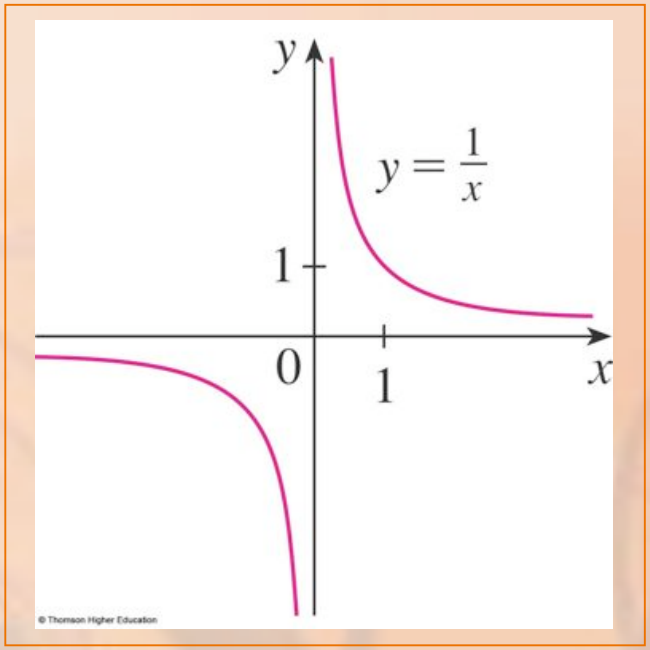
where *P* and *Q* are polynomials.

- ♣ The domain consists of all values of *x* such that $Q(x) \neq 0$.

RATIONAL FUNCTIONS

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$.

- This is the reciprocal function graphed in the figure.

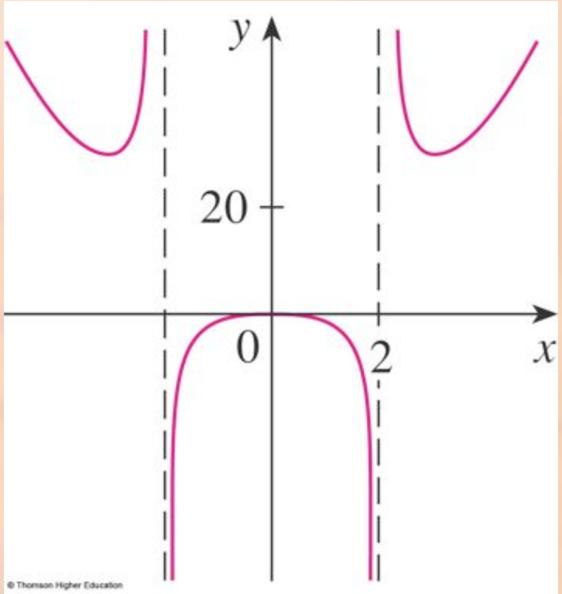


RATIONAL FUNCTIONS

The function $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$

is a rational function with domain $\{x \mid x \neq \pm 2\}$.

- Its graph is shown here.



ALGEBRAIC FUNCTIONS

A function *f* is called an algebraic function if it can be constructed using algebraic operations—such as addition, subtraction, multiplication, division, and taking roots—starting with polynomials.

ALGEBRAIC FUNCTIONS

Any rational function is automatically an algebraic function.

Here are two more examples:

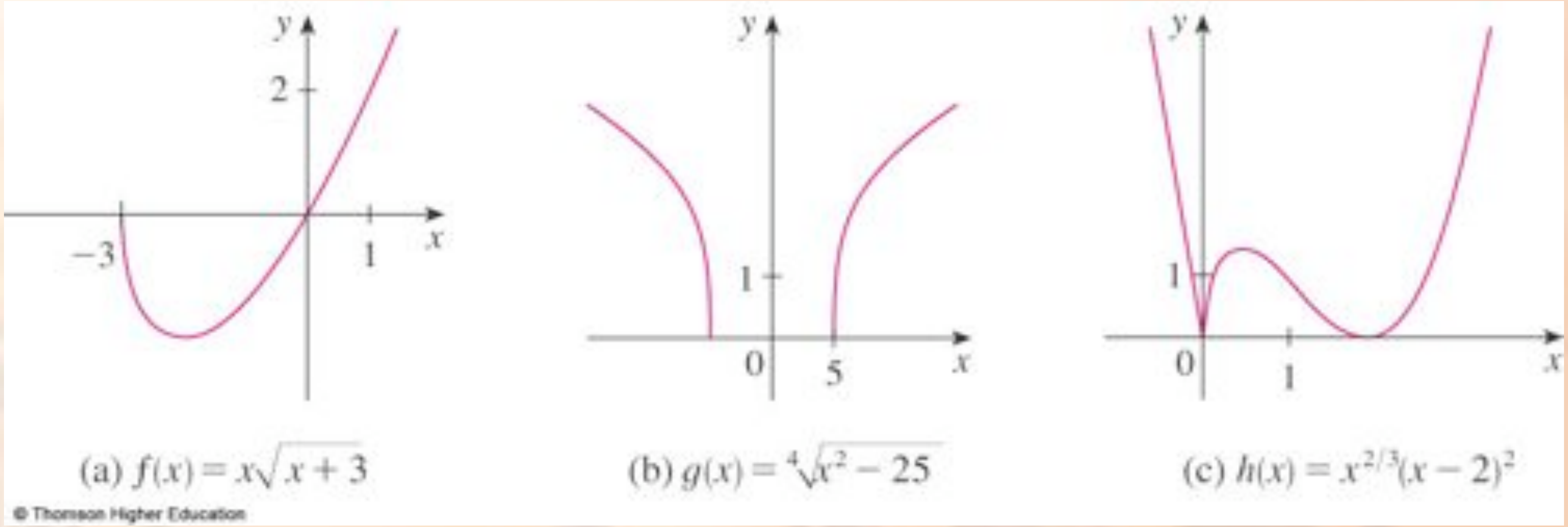
$$f(x) = \sqrt{x^2 + 1}$$

$$g(x) \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

ALGEBRAIC FUNCTIONS

When we sketch algebraic functions in Chapter 4, we will see that their graphs can assume a variety of shapes.

- ♣ The figure illustrates some of the possibilities.



An example of an algebraic function occurs in the theory of relativity.

- The mass of a particle with velocity *v* is

$$m = f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

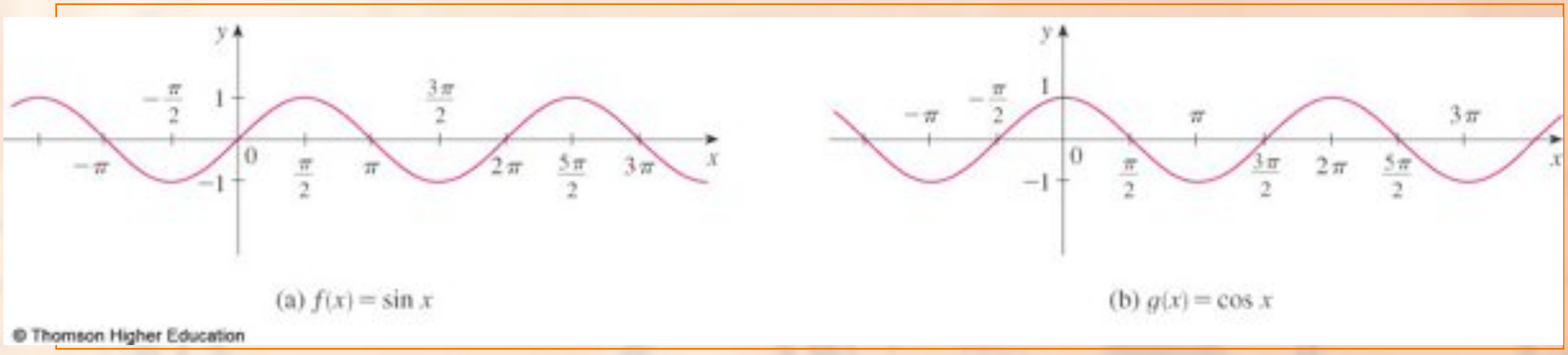
TRIGONOMETRIC FUNCTIONS

In calculus, the convention is that radian measure is always used (except when otherwise indicated).

- For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x.

TRIGONOMETRIC FUNCTIONS

- Thus, the graphs of the sine and cosine functions are as shown in the figure.



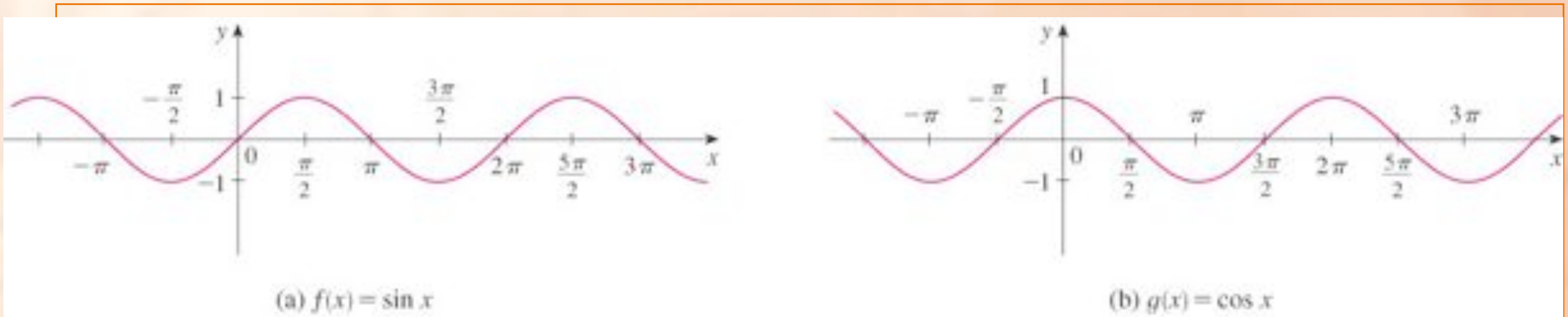
TRIGONOMETRIC FUNCTIONS

Notice that, for both the sine and cosine functions, the domain is $(-\infty, \infty)$ and the range is the closed interval [-1, 1].

- Thus, for all values of *x*, we have:

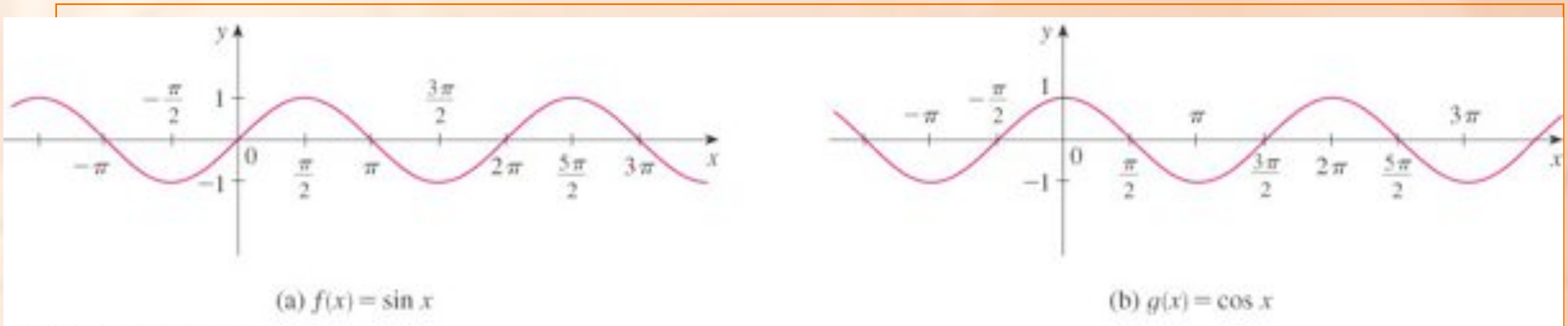
  $-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1$

- In terms of absolute values, it is: $|\sin x| \leq 1 \qquad |\cos x| \leq 1$



Also, the zeros of the sine function occur at the integer multiples of π.

That is, $\sin x = 0$ when $x = n\pi$, n an integer.

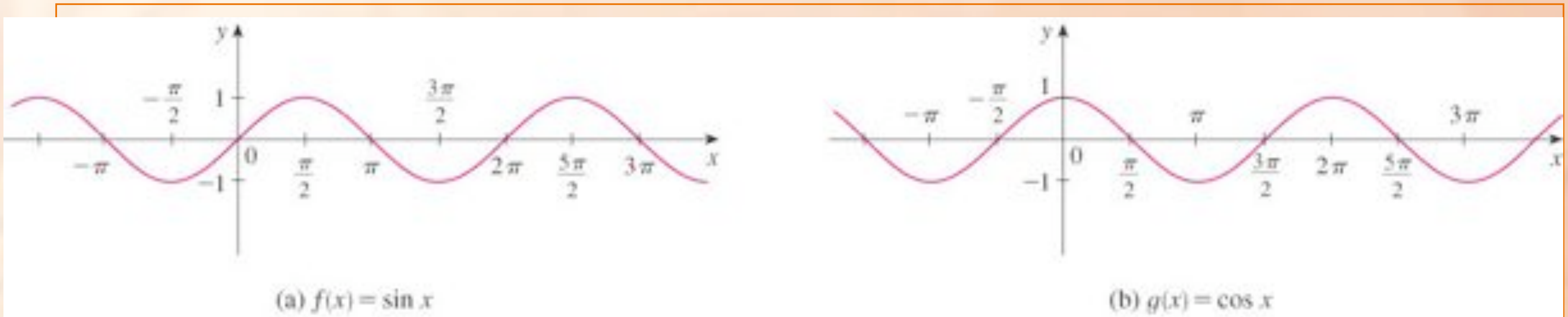


TRIGONOMETRIC FUNCTIONS

An important property of the sine and cosine functions is that they are periodic functions and have a period 2π.

- This means that, for all values of *x*,

$$\sin(x+2\pi)=\sin x \qquad \cos(x+2\pi)=\cos x$$



TRIGONOMETRIC FUNCTIONS

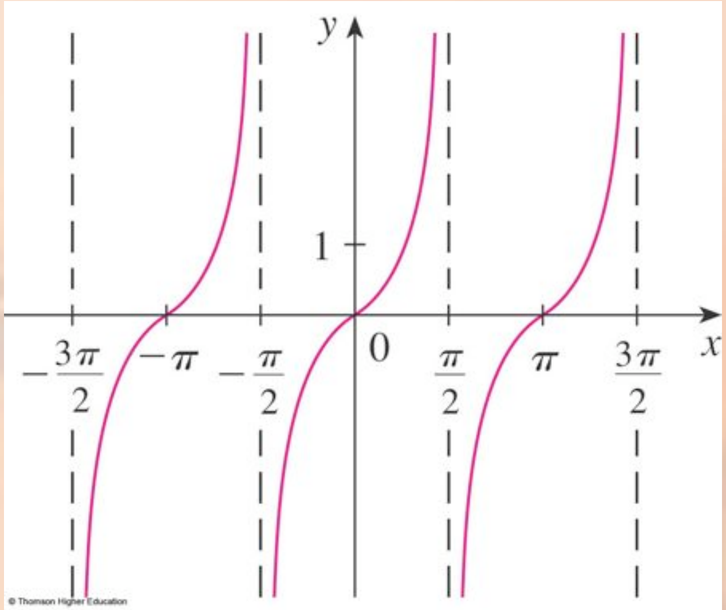
The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves.

For instance, in Example 4 in Section 1.3, we will see that a reasonable model for the number of hours of daylight in Philadelphia *t* days after January 1 is given by the function:

$$L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t-80)\right]$$

The tangent function is related to the sine and cosine functions by the equation $\tan x = \dfrac{\sin x}{\cos x}$

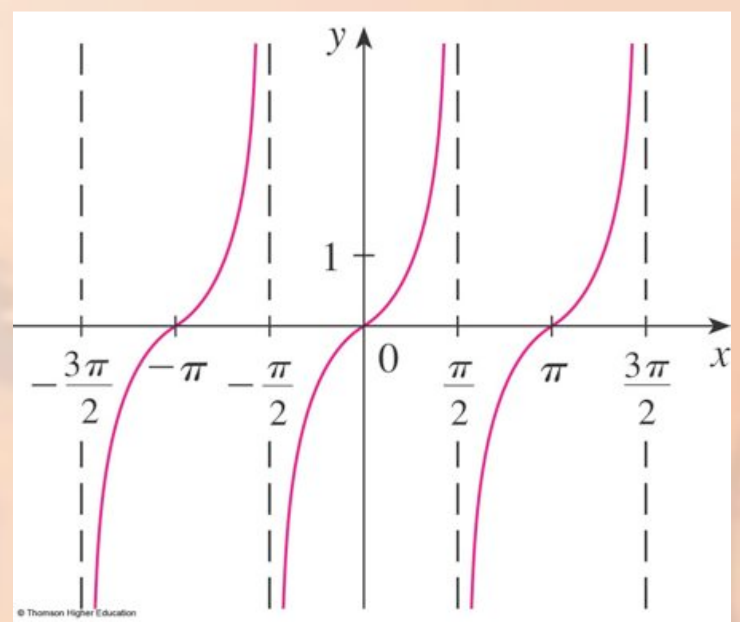
Its graph is shown.



TRIGONOMETRIC FUNCTIONS

The tangent function is undefined whenever $\cos x = 0$, that is, when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$.
Its range is $(-\infty, \infty)$.

- Notice that the tangent function has period π:
  $\tan(x+\pi) = \tan x$ for all x.



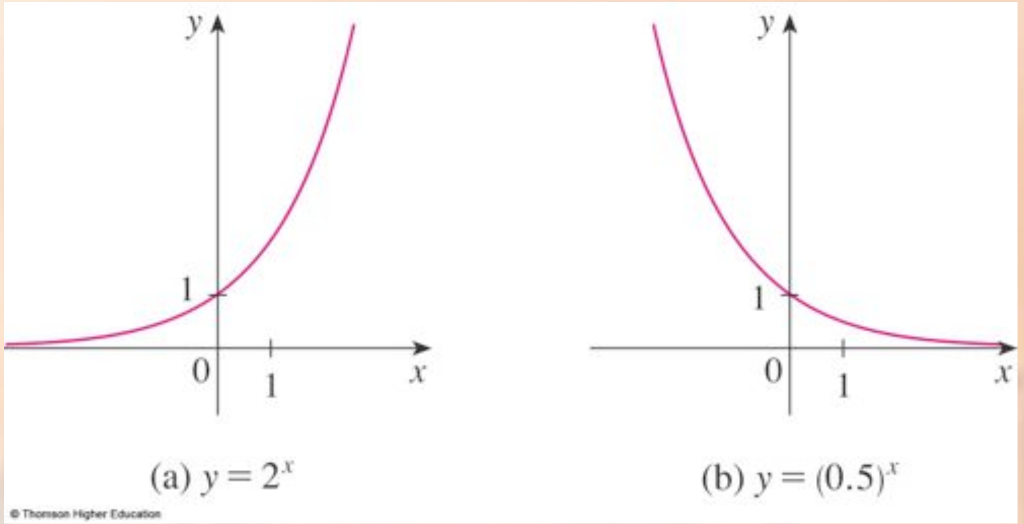
TRIGONOMETRIC FUNCTIONS

The remaining three trigonometric functions—cosecant, secant, and cotangent—are the reciprocals of the sine, cosine, and tangent functions.

EXPONENTIAL FUNCTIONS

The exponential functions are the functions of the form $f(x) = a^x$, where the base *a* is a positive constant.

- The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown.
- In both cases, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



We will study exponential functions in detail in Section 1.5.

- We will see that they are useful for modeling many natural phenomena—such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

LOGARITHMIC FUNCTIONS

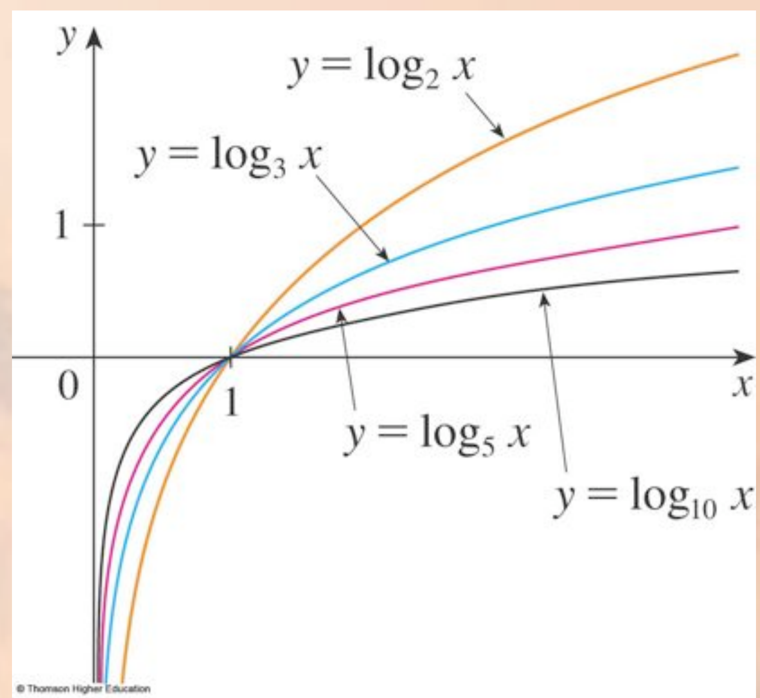
The logarithmic functions $f(x) = \log_a x$, where the base *a* is a positive constant, are the inverse functions of the exponential functions.

- ♣ We will study them in Section 1.6.

LOGARITHMIC FUNCTIONS

The figure shows the graphs of four logarithmic functions with various bases.

- In each case, the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when x > 1.



TRANSCENDENTAL FUNCTIONS

Transcendental functions are those that are not algebraic.

- The set of transcendental functions includes the trigonometric, inverse trigonometric, exponential, and logarithmic functions.
- However, it also includes a vast number of other functions that have never been named.
- In Chapter 11, we will study transcendental functions that are defined as sums of infinite series.

 Example 5

Classify the following functions as one of the types of functions that we have discussed.

a. $f(x) = 5^x$

b. $g(x) = x^5$

c. $h(x) = \dfrac{1+x}{1-\sqrt{x}}$

d. $u(t) + 1 - t + 5t^4$

Example 5 a

$f(x) = 5^x$ is an exponential function.

- The *x* is the exponent.

Example 5 b

$g(x) = x^5$ is a power function.

- The *x* is the base.

We could also consider it to be a polynomial of degree 5.

Example 5 c

$$h(x) = \frac{1+x}{1-\sqrt{x}}$$ is

an algebraic function.

Example 5 d

$u(t) = 1 - t + 5t^4$ is

a polynomial of degree 4.