6

APPLICATIONS OF INTEGRATION

In this chapter, we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, and the work done by a varying force.

- The common theme is the following general method—which is similar to the one used to find areas under curves.

We break up a quantity *Q* into a large number of small parts.

- Next, we approximate each small part by a quantity of the form $f(x_i^*)\Delta x$ and thus approximate *Q* by a Riemann sum.
- Then, we take the limit and express *Q* as an integral.
- Finally, we evaluate the integral using the Fundamental Theorem of Calculus or the Midpoint Rule.

6.1
Areas Between Curves

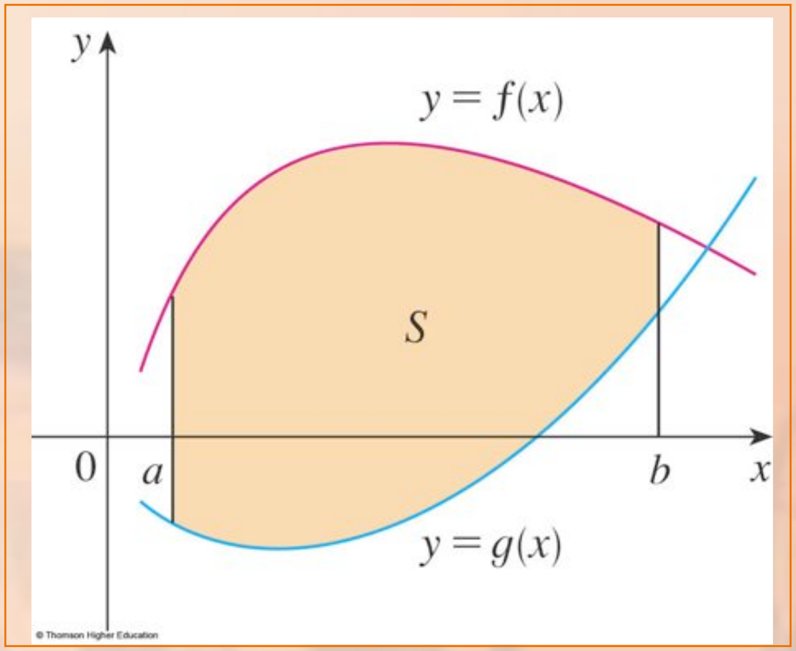
In this section we learn about:

Using integrals to find areas of regions that lie between the graphs of two functions.

AREAS BETWEEN CURVES

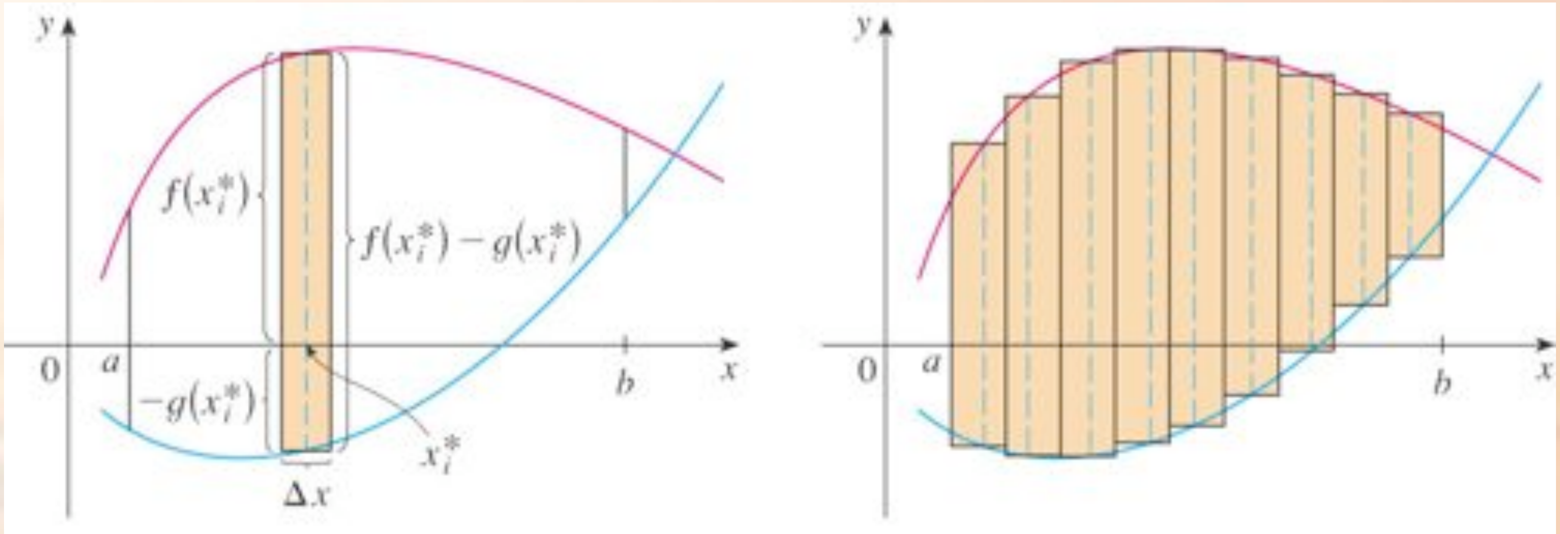
Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.

- Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.



AREAS BETWEEN CURVES

As we did for areas under curves in Section 5.1, we divide *S* into *n* strips of equal width and approximate the *i* th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$.

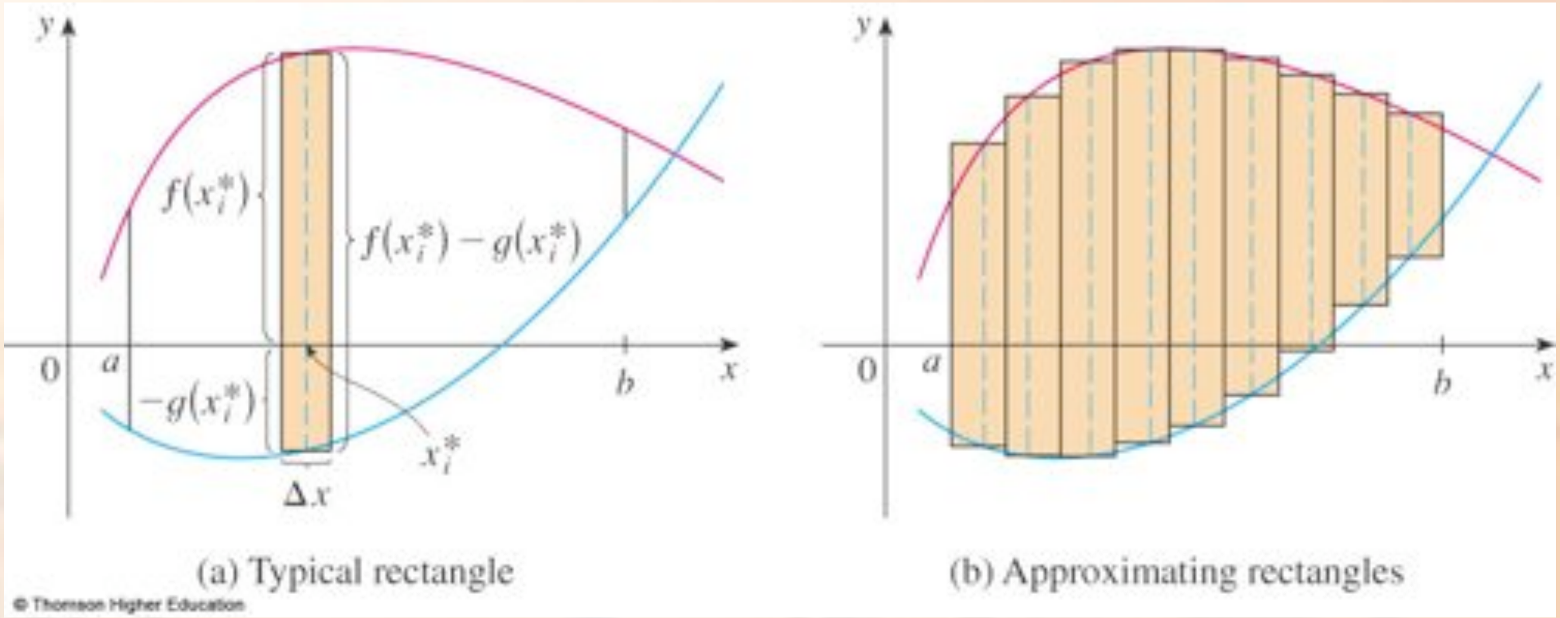


(a) Typical rectangle

(b) Approximating rectangles

AREAS BETWEEN CURVES

We could also take all the sample points to be right endpoints—in which case $x_i^* = x_i$.



The Riemann sum $\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\,\Delta x$

is therefore an approximation to what we intuitively think of as the area of *S*.

- This approximation appears to become better and better as n _ ∞.

Definition 1

Thus, we define the area *A* of the region *S* as the limiting value of the sum of the areas of these approximating rectangles.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x$$

♣ The limit here is the definite integral of *f* - *g*.

Thus, we have the following formula for area:

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

AREAS BETWEEN CURVES

Notice that, in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area reduces to Definition 2 in Section 5.1

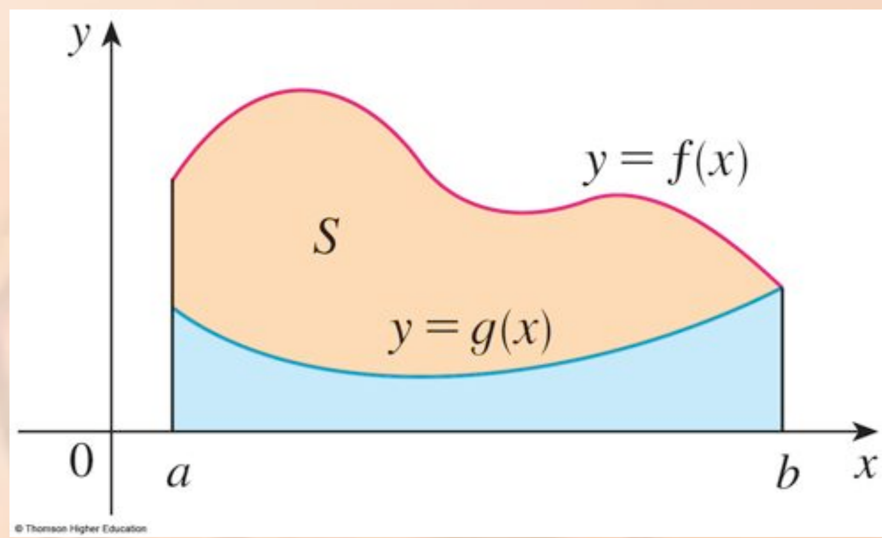
AREAS BETWEEN CURVES

Where both *f* and *g* are positive, you can see from the figure why Definition 2 is true:

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\, dx$$

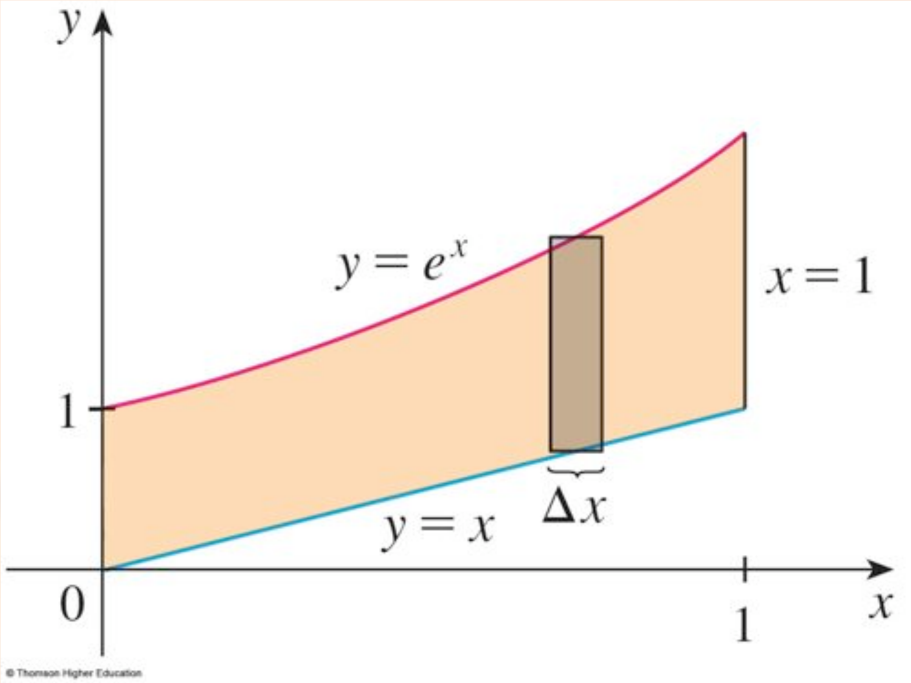


Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Example 1

As shown here, the upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$.



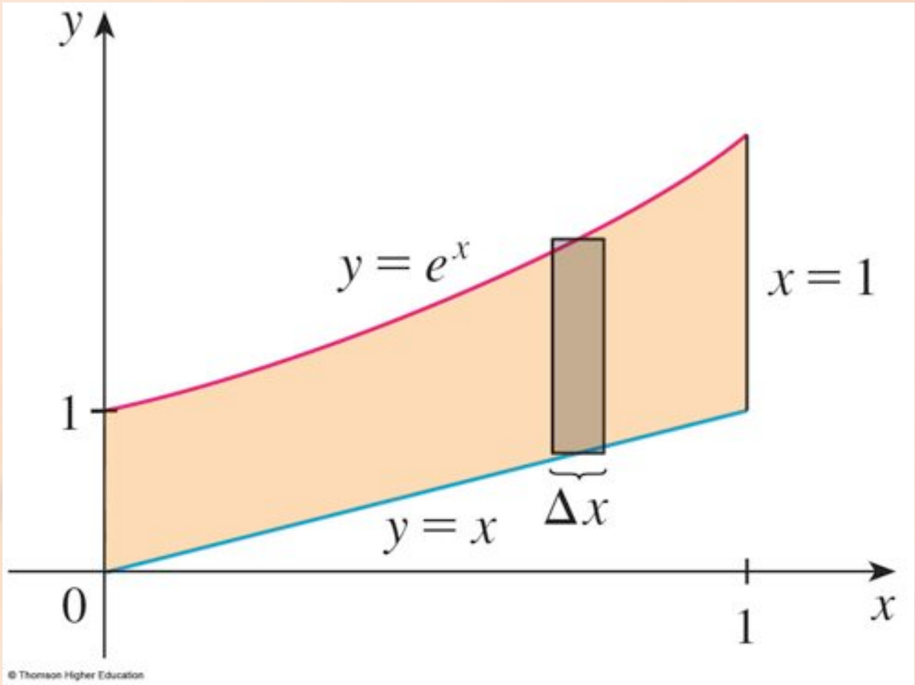
So, we use the area formula with $y = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 \left(e^x - x\right) dx = e^x - \tfrac{1}{2}x^2 \Big]_0^1$$

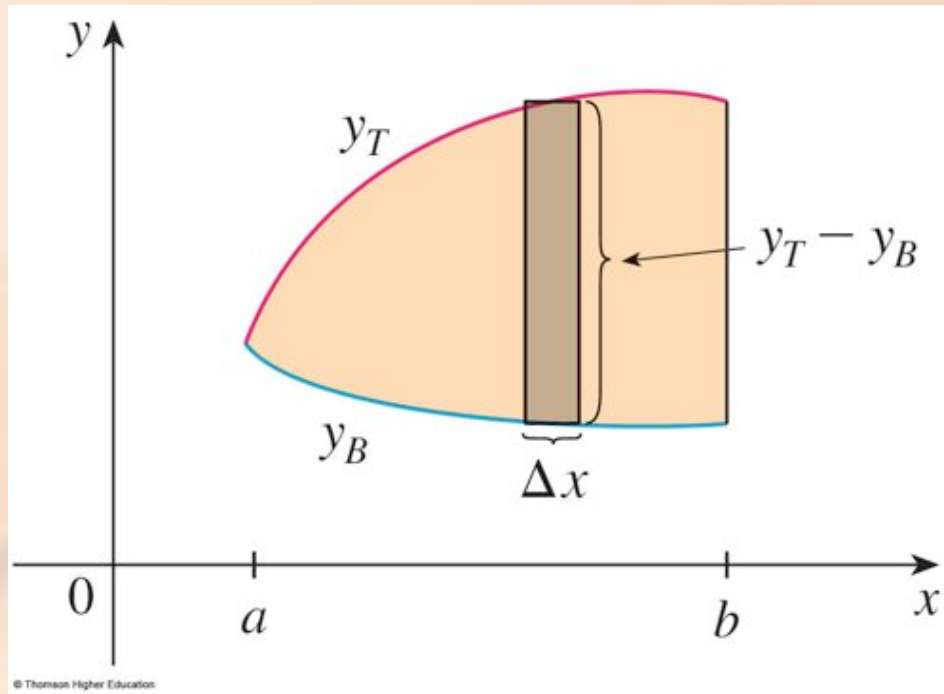
$$= e - \frac{1}{2} - 1 = e - 1.5$$

AREAS BETWEEN CURVES

Here, we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in Definition 1.



In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle.



AREAS BETWEEN CURVES

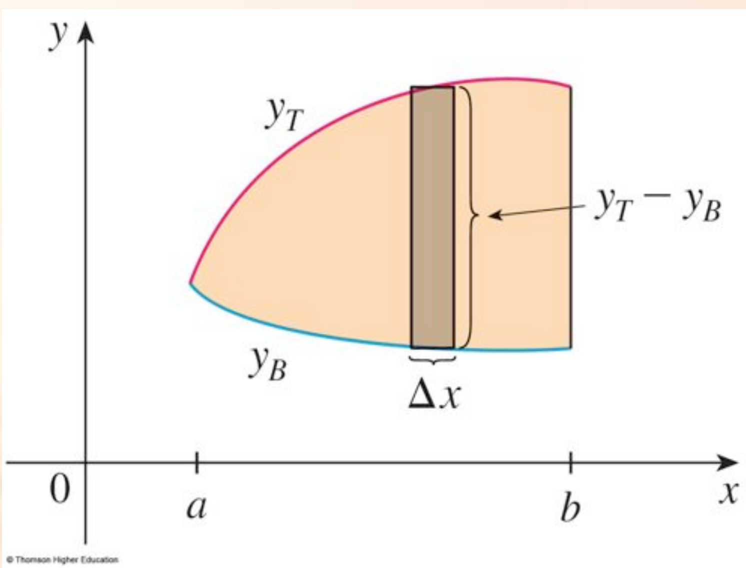
Then, the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

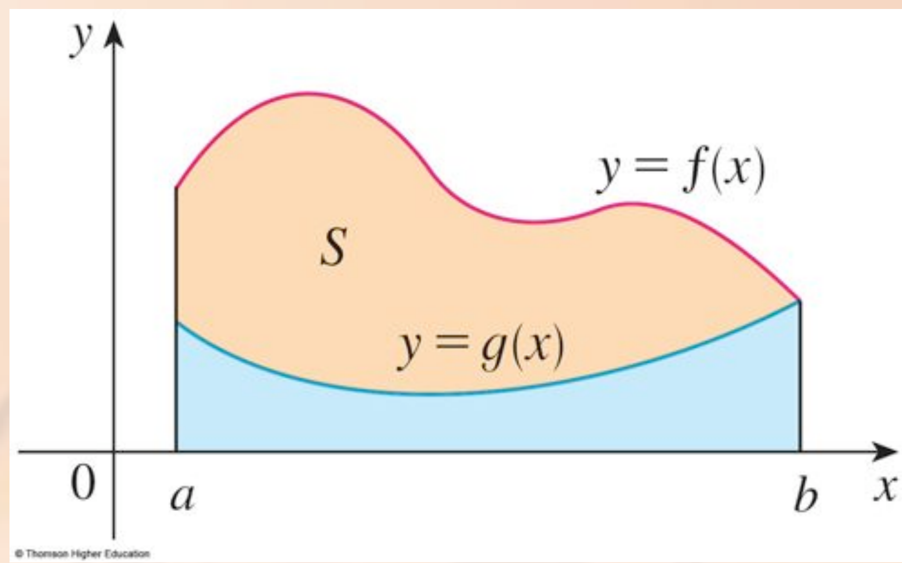
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (y_T - y_B)\Delta x = \int_a^b (y_T - y_B)\, dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

AREAS BETWEEN CURVES

Notice that, in the first figure, the left-hand boundary reduces to a point whereas, in the other figure, the right-hand boundary reduces to a point.





In the next example, both the side boundaries reduce to a point.

- ♣So, the first step is to find *a* and *b*.

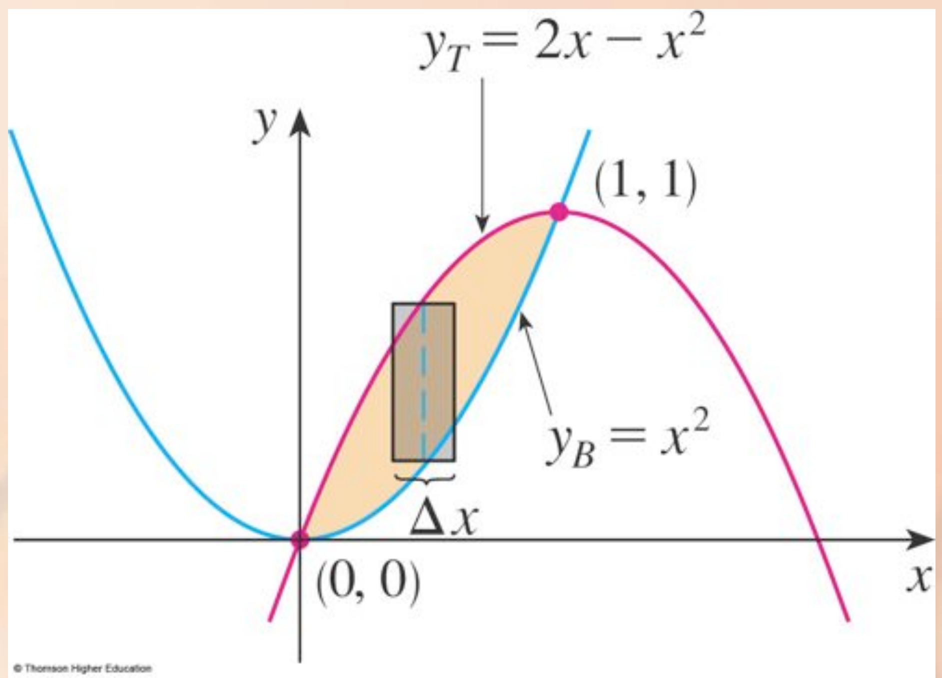
Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

First, we find the points of intersection of the parabolas by solving their equations simultaneously.

- This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$.
- Thus, $2x(x - 1) = 0$, so $x = 0$ or 1.
- The points of intersection are (0, 0) and (1, 1).

From the figure, we see that the top and bottom boundaries are:

$y_T = 2x - x^2$ and $y_B = x^2$



The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$.

- So, the total area is:

$$A = \int_0^1 \left(2x - 2x^2\right) dx = 2\int_0^1 \left(x - x^2\right) dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

AREAS BETWEEN CURVES

Sometimes, it is difficult—or even impossible—to find the points of intersection of two curves exactly.

- As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

If we were to try to find the exact intersection points, we would have to solve the equation

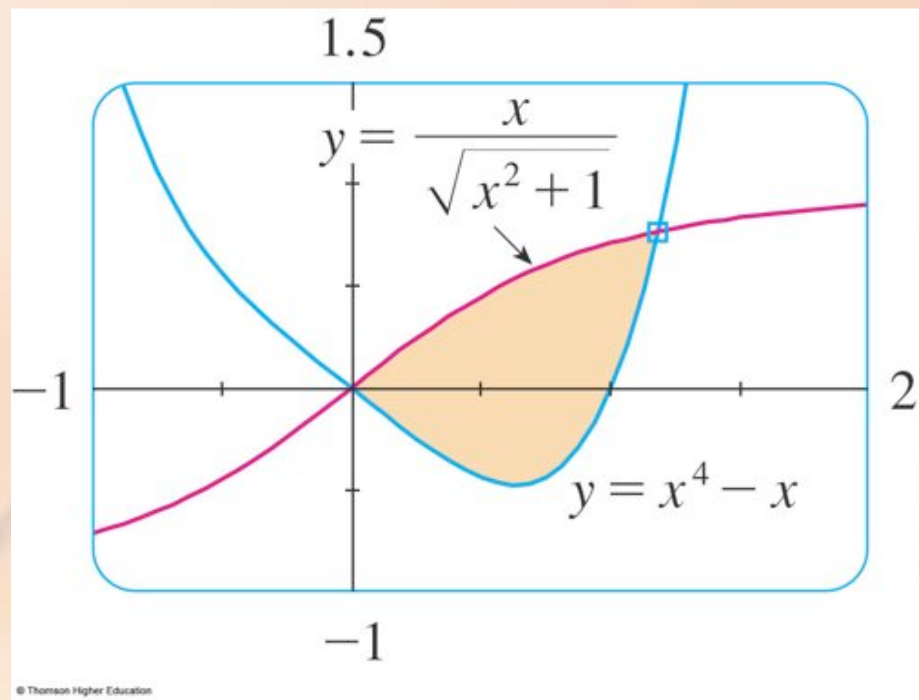
$$\frac{x}{\sqrt{x^2+1}} = x^4 - x$$

- It looks like a very difficult equation to solve exactly.
- In fact, it's impossible.

 Example 3

Instead, we use a graphing device to draw the graphs of the two curves.

- One intersection point is the origin. The other is $x \approx 1.18$
- If greater accuracy is required, we could use Newton's method or a rootfinder—if available on our graphing device.



Thus, an approximation to the area between the curves is:

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2+1}} - \left(x^4 - x \right) \right] dx$$

- To integrate the first term, we use the substitution $u = x^2 + 1$.
- Then, $du = 2x\,dx$, and when $x = 1.18$, we have $u \approx 2.39$

Therefore,

$$A \approx \tfrac{1}{2}\int_1^{2.39} \frac{du}{\sqrt{u}} - \int_0^{1.18} \left(x^4 - x\right)dx$$

$$= \sqrt{u}\,\Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2}\right]_0^{1.18}$$

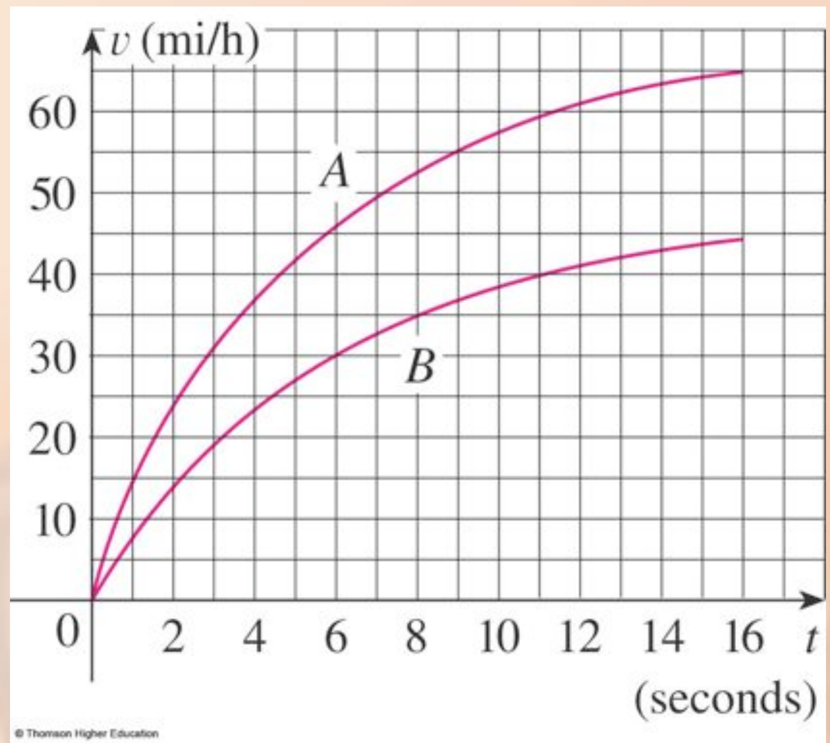
$$= \sqrt{2.39} - 1 - \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2}$$

$$\approx 0.785$$

The figure shows velocity curves for two cars, A and B, that start side by side and move along the same road.

What does the area between the curves represent?

- Use the Midpoint Rule to estimate it.

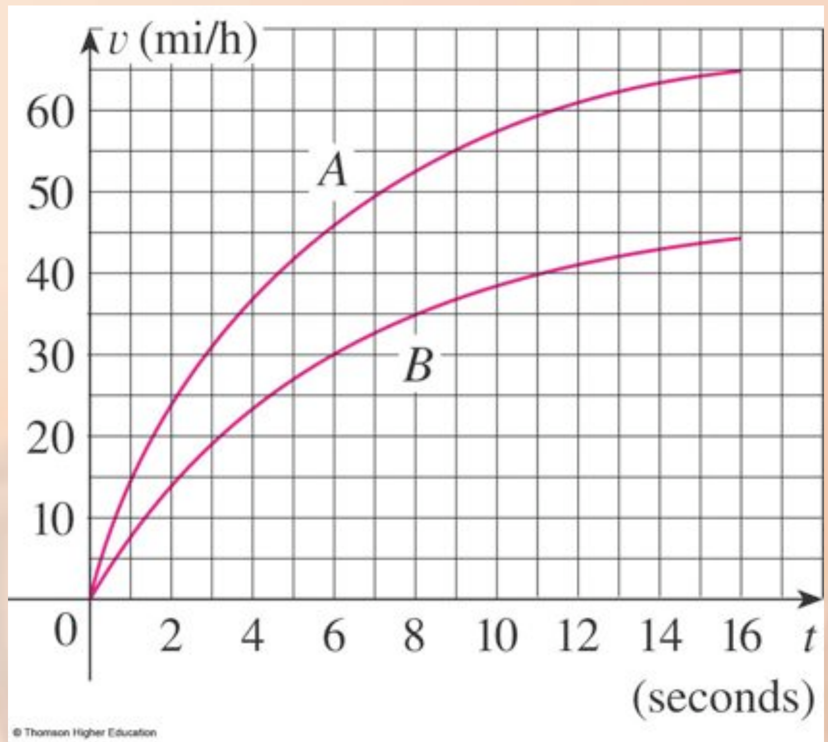


AREAS BETWEEN CURVES

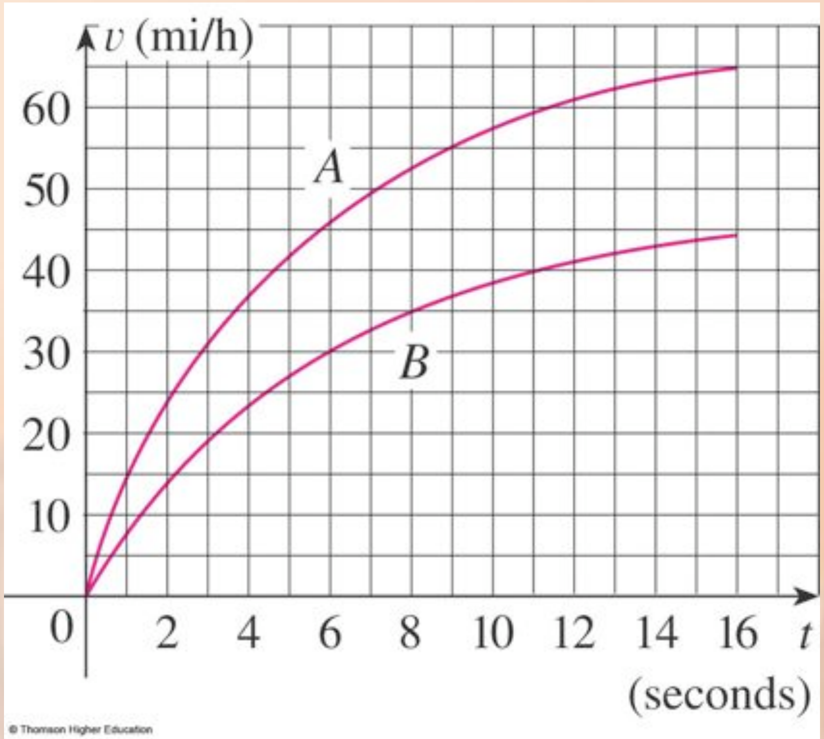
Example 4

The area under the velocity curve *A* represents the distance traveled by car A during the first 16 seconds.

- Similarly, the area under curve *B* is the distance traveled by car B during that time period.



So, the area between these curves—which is the difference of the areas under the curves—is the distance between the cars after 16 seconds.

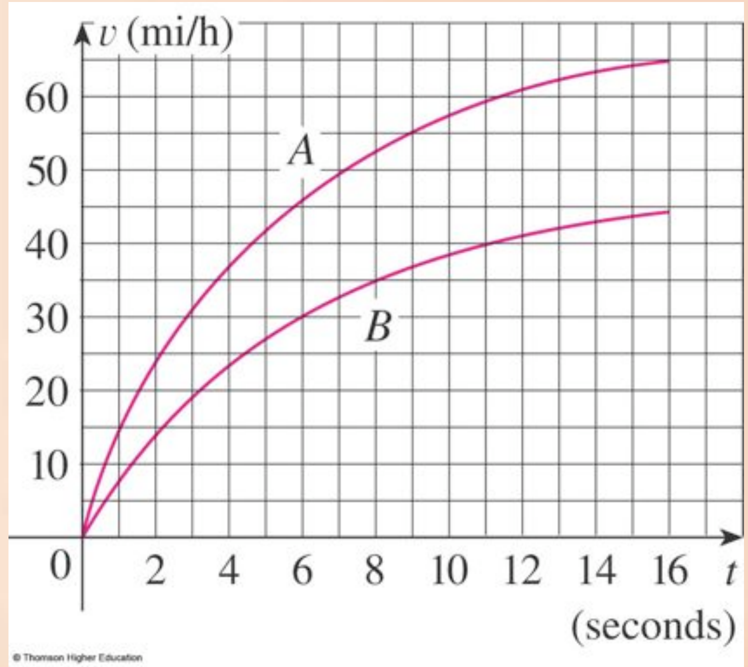


AREAS BETWEEN CURVES

Example 4

We read the velocities from the graph and convert them to feet per second

$$\left(1\,\text{mi/h} = \frac{5280}{3600}\,\text{ft/s}\right)$$



t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - v_B$	0	13	20	23	25	28	29	29	30

We use the Midpoint Rule with *n* = 4 intervals, so that $\Delta t = 4$.

- The midpoints of the intervals are $\overline{t_1} = 2$, $\overline{t_2} = 6$, $\overline{t_3} = 10$, and $\overline{t_4} = 14$.

We estimate the distance between the cars after 16 seconds as follows:

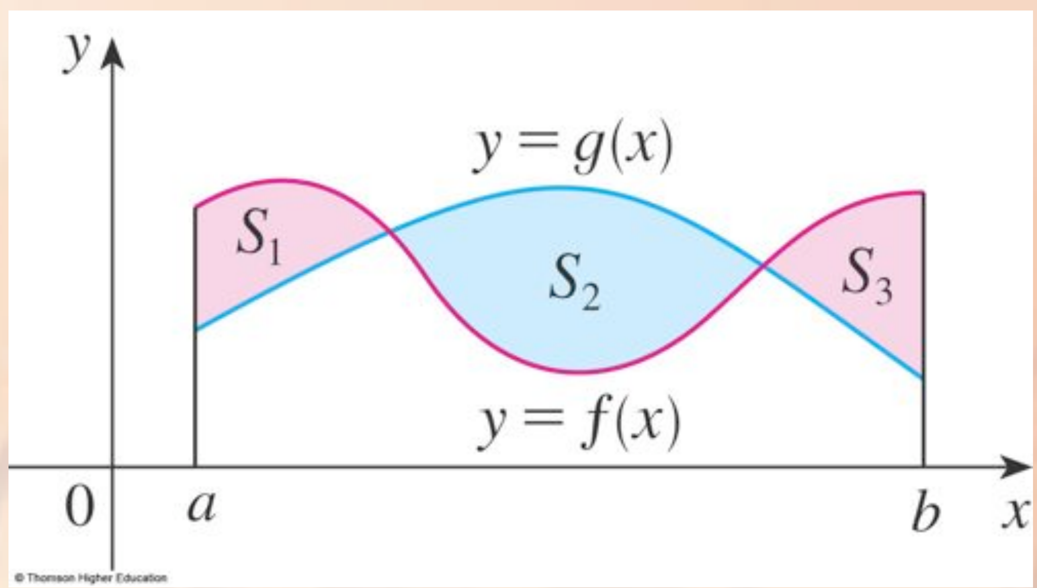
$$\int_0^{16} (v_A - v_B)\, dt \approx \Delta t\left[13 + 23 + 28 + 29\right]$$

$$= 4(93)$$

$$= 372 \text{ ft}$$

AREAS BETWEEN CURVES

To find the area between the curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, split the given region S into several regions S_1, S_2, . . . with areas A_1, A_2, . . .



Then, we define the area of the region S to be the sum of the areas of the smaller regions S_1, S_2, . . . , that is, $A = A_1 + A_2 + \ldots$

Since

$$\left|f(x)-g(x)\right| = \begin{cases} f(x)-g(x) & \text{when } f(x)\geq g(x) \\ g(x)-f(x) & \text{when } g(x)\geq f(x) \end{cases}$$

we have the following expression for *A*.

Definition 3

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is:

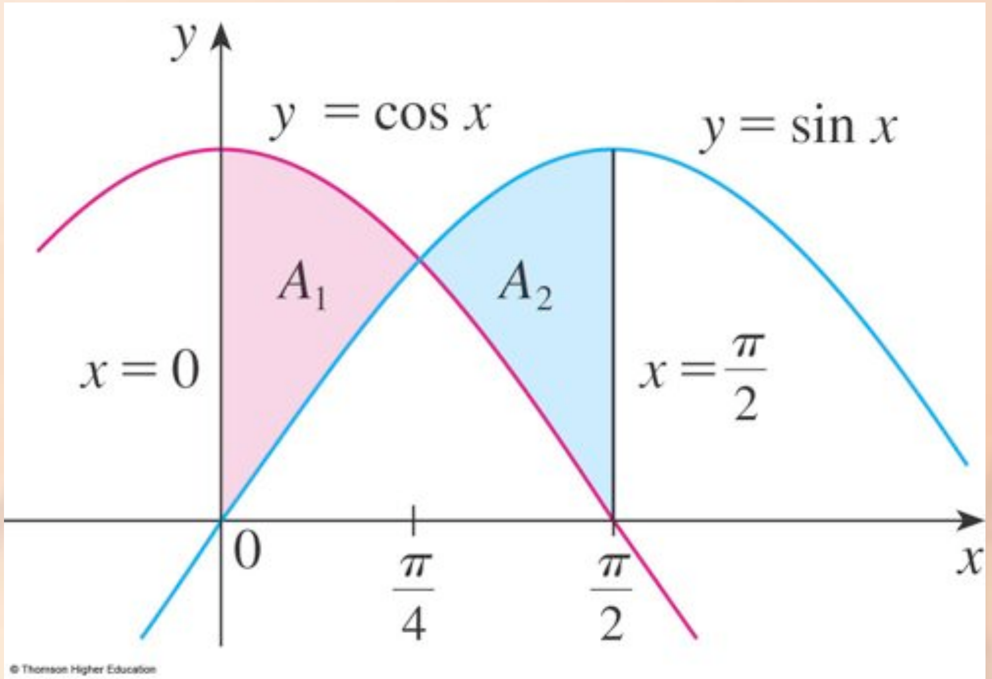
$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

- However, when evaluating the integral, we must still split it into integrals corresponding to A_1, A_2, ...

Example 5

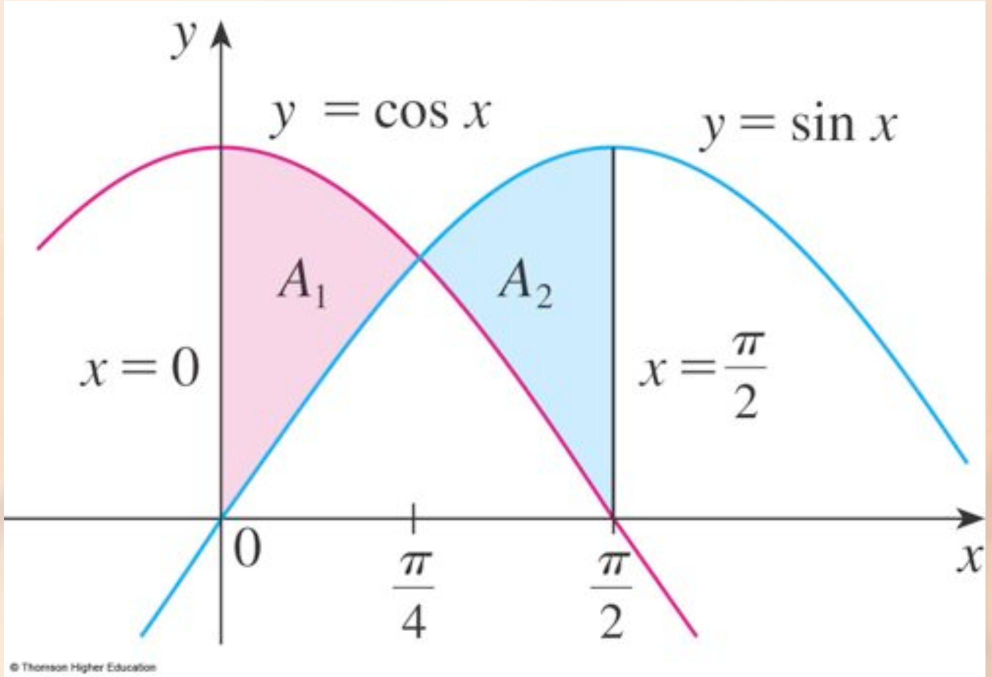
Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi / 4$ (since $0 \le x \le \pi / 2$).



Example 5

Observe that $\cos x \geq \sin x$ when $0 \leq x \leq \pi / 4$ but $\sin x \geq \cos x$ when $\pi / 4 \leq x \leq \pi / 2$.



So, the required area is:

$$A = \int_0^{\pi/2} \left|\cos x - \sin x\right| \, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}$$

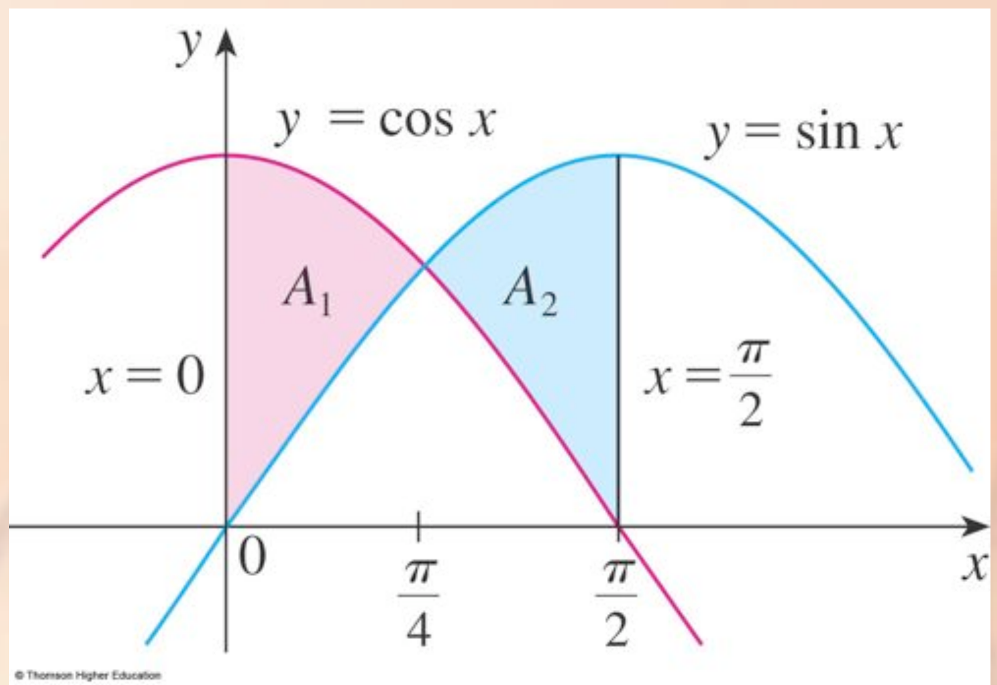
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)$$

$$= 2\sqrt{2} - 2$$

 Example 5

We could have saved some work by noticing that the region is symmetric about $x = \pi / 4$.

So, $A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$

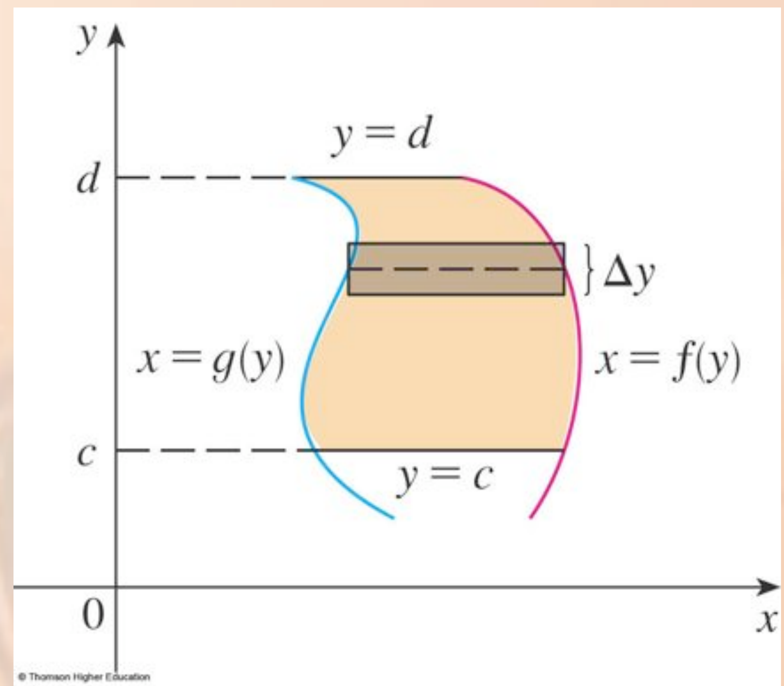


AREAS BETWEEN CURVES

Some regions are best treated by regarding *x* as a function of *y*.

- If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where *f* and *g* are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is:

$$A = \int_c^d [f(y) - g(y)]\,dy$$

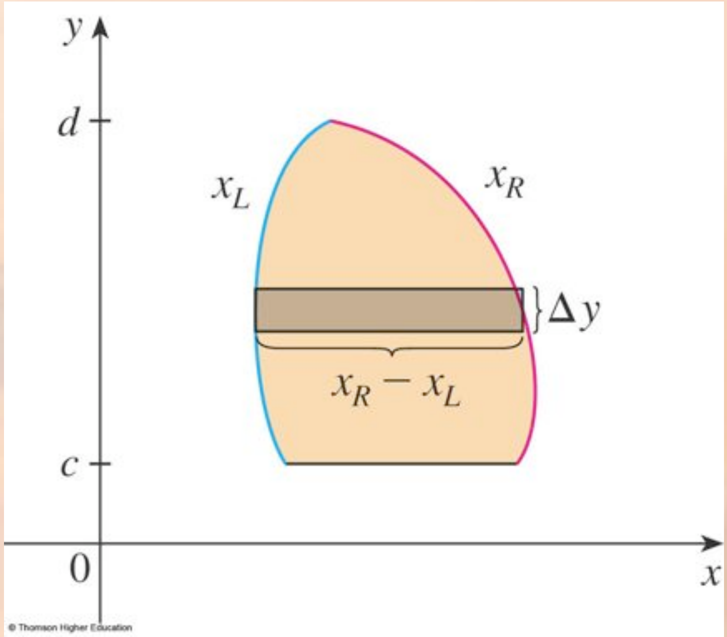


AREAS BETWEEN CURVES

If we write x_R for the right boundary and x_L for the left boundary, we have:

$$A = \int_c^d (x_R - x_L)\,dy$$

- Here, a typical approximating rectangle has dimensions x_R - x_L and Δy.



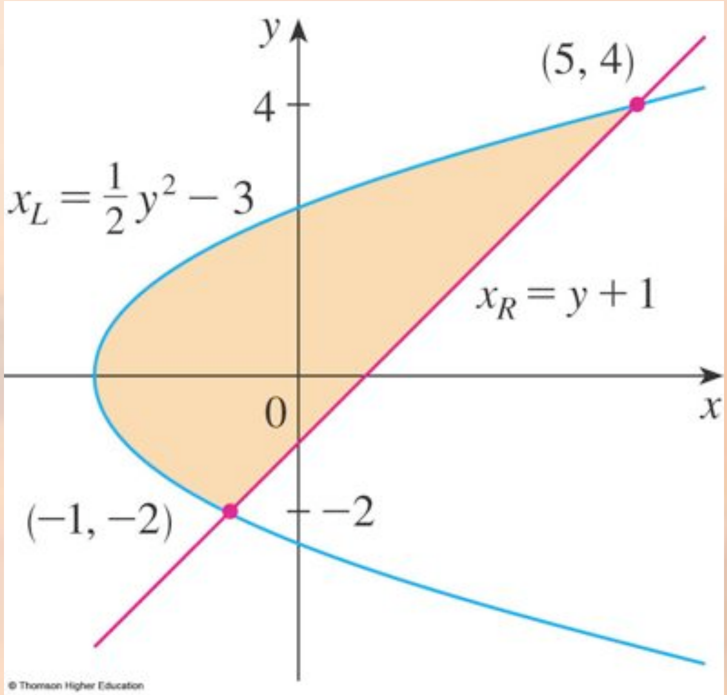
Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

By solving the two equations, we find that the points of intersection are (-1, -2) and (5, 4).

- We solve the equation of the parabola for *x.*
- From the figure, we notice that the left and right boundary curves are:

$$x_L = \frac{1}{2}y^2 - 3$$

$$x_R = y + 1$$



We must integrate between the appropriate y-values, $y = -2$ and $y = 4$.

Thus, $A = \int_{-2}^{4} (x_R - x_L)\, dy$

$$= \int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2} y^2 - 3 \right) \right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2} y^2 + y + 4 \right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3} \right) + \frac{y^2}{2} + 4y \Big]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8 \right) = 18$$

In the example, we could have found the area by integrating with respect to x instead of y.

However, the calculation is much more involved.

It would have meant splitting the region in two and computing the areas labeled A_1 and A_2.

- The method used in the Example is much easier.