MTH 120 — Spring — 2008
Essex County College — Division of Mathematics
Handout — Version 13[1] — February 14, 2008

1 Complex Numbers

Attached to this short introduction is a rather lengthy explanation of why complex numbers are so important. Fact is, you were probably told that the unit imaginary number is necessary mainly because we need them to solve some quadratic equations. However, the fact remains that their rise to importance is really not due to quadratics but more to do with an Italian named Gerolamo Cardano, and he is remembered here because he showed that complex numbers were absolutely necessary in solving a certain cubic equation.

As I hope you recall the unit imaginary is denoted by i, where

$$i = \sqrt{-1}.$$

You should further recall that a complex number is written as a linear combination of a real and imaginary part, and is usually denoted by the letter z, where

$$z = a + bi.$$

We're are going to visualize these complex numbers as an ordered pair $(a,\ b)$. Here a is the position along the real axis (similar to the x-axis) and b is the position along the imaginary axis (similar to the y-axis).

Here if we visualize the point $(a,\ b)$, we can easily find many relationships to trigonometry and geometry. For example, in class we will illustrate the following relationships.

1. The **modulus** (or absolute value) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

2. The complex number $z = a + bi$ has a **trigonometric form** (or **polar form**)

$$z = r\left(\cos\theta + i\sin\theta\right)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

3. If two complex number z_1 and z_2 have polar forms

$$z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right) \quad \text{and} \quad z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$$

then

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right] && \text{multiplication} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right],\ z_2 \neq 0 && \text{division} \end{aligned}$$

[1]This document was prepared by Ron Bannon using LaTeX 2ε.

Let's take some examples.

1. Find the modulus of the complex number $3 + 4i$.

2. Find the modulus of the complex number $8 - 5i$.

3. Write the given complex number in trigonometric form.
$$1 + i$$

4. Write the given complex number in trigonometric form.
$$-4\sqrt{3} - 4i$$

5. Write the given trigonometric form of a complex number in the form $a + bi$.
$$2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

6. Write the given trigonometric form of a complex number in the form $a + bi$.
$$5\left[\cos\left(\arctan\frac{4}{3}\right) + i\sin\left(\arctan\frac{4}{3}\right)\right]$$

7. Let
$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right),$$
find both $z_1 z_2$ and z_1/z_2.

8. Repeat the last problem using the form $a + bi$ and verify that it is the same.

9. **Optional:** Read the article that follows. It's not easy, but you may learn why complex numbers proved unavoidable, and it has nothing to do with quadratics.

2 DeMoivre's Theorem

If we were to take our multiplication rule, where $z_1 = z_2 = z$, we would get:

$$\begin{aligned} zz &= rr\left[\cos\left(\theta + \theta\right) + i\sin\left(\theta + \theta\right)\right]; \\ z^2 &= r^2\left[\cos 2\theta + i\sin 2\theta\right]. \end{aligned}$$

Of course if we keep repeating this n times for the same z we would get:

$$z^n = r^n\left[\cos n\theta_1 + i\sin n\theta_1\right].$$

Basically DeMoivre's Theorem tells us that to take the n^{th} of any complex number, we take the n^{th} power of the *modulus* and multiple the *argument* by n.

In MTH-100 you were asked to raise complex numbers to integral powers, and this was generally easy to do if the powers were small. For example:

$$(2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i.$$

However, suppose you were asked to do

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}?$$

Multiplying this out, even if you are familiar with expanding binomials, would be difficult for most students. Using DeMoivre's Theorem makes it a lot simpler. First write in *polar* form

$$\frac{1}{2}+\frac{1}{2}i=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),$$

then we have

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}=\left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right).$$

If you simplify you'll find:

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}=\frac{1}{32}\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)=\frac{1}{32}i.$$

We can extend this to take n^{th} roots of a complex number by letting $w=\sqrt[n]{z}$ or $w^n=z$. To find the n^{th} root of z, we basically need to find a complex number w such that $w^n=z$. If

$$z=r\left(\cos\theta+i\sin\theta\right),$$

then

$$w=\sqrt[n]{r}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)$$

However, you may have noticed that the argument θ can be replaced by any $\theta+2\pi k$, $k\in\mathbb{Z}$. So we have

$$w=\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right)$$

where $k=0,\ 1,2,\ \ldots n-1$.

As an example, let's find the six sixth roots of $z=-64$. Here's the steps:

1. Write the polar form of z.

$$z=64\left(\cos\pi+i\sin\pi\right)$$

2. Use the formula.

$$w_k=\sqrt[6]{64}\left(\cos\frac{\pi+2\pi k}{6}+i\sin\frac{\pi+2\pi k}{6}\right)$$

3. Now compute.

$$\begin{aligned}
w_0 &= \sqrt[6]{64}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\sqrt{3}+i\\
w_1 &= \sqrt[6]{64}\left(\cos\frac{\pi+2\pi}{6}+i\sin\frac{\pi+2\pi}{6}\right)=2i\\
w_2 &= \sqrt[6]{64}\left(\cos\frac{\pi+4\pi}{6}+i\sin\frac{\pi+4\pi}{6}\right)=-\sqrt{3}+i\\
w_3 &= \sqrt[6]{64}\left(\cos\frac{\pi+6\pi}{6}+i\sin\frac{\pi+6\pi}{6}\right)=-\sqrt{3}-i\\
w_4 &= \sqrt[6]{64}\left(\cos\frac{\pi+8\pi}{6}+i\sin\frac{\pi+8\pi}{6}\right)=-2i\\
w_5 &= \sqrt[6]{64}\left(\cos\frac{\pi+10\pi}{6}+i\sin\frac{\pi+10\pi}{6}\right)=\sqrt{3}-i
\end{aligned}$$

If were dealing with real numbers, the solution to

$$x^6 + 64 = 0$$

would be the empty set. However, if you dealing with complex number the solution to this same equation would be

$$\left\{\sqrt{3} + i,\ 2i,\ -\sqrt{3} + i,\ -\sqrt{3} - i,\ -2i,\ \sqrt{3} - i\right\}.$$

This is exactly what we learned in MTH-119 (there's six roots).

Now for some examples to work on.

1. Find the three cube roots of $z = 2 + 2i$.

2. Verify using polar form that i^9 is i.

3. Find the three cube roots of 1.

4. Solve for x, where x is complex.

   (a) $x^5 - 1 = 0$

   (b) $x^4 + 81 = 0$

(c) $x^5 + \sqrt{3} + i = 0$

3 The Cubic and Its Lost Role in Education . . .

I think it would be an utter and complete surprise to my high-school teachers that I actually became a teacher, and even more disturbing to them would be the fact that I'm a mathematics teacher. For one, I am incredibly shy and was just an average student in mathematics, but more important is that when I left high-school I followed a more *artistic* path that was devoid of rigor, mathematical or otherwise. Time has a meandering way of making the impossible happen and I eventually found my way—pure accident—into a teaching job[2] where I was confronted with the responsibility of judging the mathematical mettle of those under my charge. Well, I've been teaching now, almost continuously since 1988 and will continue teaching for as long as I can, or until the 'they' in administration push me out. My meandering path has become more settled, but I nonetheless try to look beyond what I teach to see if I can do better, and so this short paper is about doing better as a teacher.

My preparation to teach mathematics was not thorough and although I have extensive academic experience, I am not a mathematician by any standard. However, I can certainly recall much of the basics that I've learned from the introductory mathematics books that I've read, and more importantly the tidbits of advice that my better educated mathematics teachers shared with me. One teacher in particular was a well-educated PhD from Columbia University that lamented the fact that the solutions to cubics are no longer being taught. That tidbit has stayed with me for more than I care to remember, and I doubt he'd want to be remembered for that alone, but it those events in my education that weren't necessarily planned that remain with me long after I've forgotten the more arcane details. To be frank, this man had many unusual characteristics that left an indelible mark in my mind—many of which had nothing to do with mathematics, but rather to what he'd referred to as being educated. We, the students that is, sat quietly and were constantly hoping that he wouldn't ask about all the minutia that was being presented; but we were nonetheless in awe of this man's ability to *know* the grand mathematical picture . . . at least I believed his peculiar nature somehow entitled him increased intellectual reverence.

As a teacher I tend to repeat what I've heard as a student, and this particular teacher has provided much fodder for later regurgitation. For example, when I teach precalculus, we are required to teach factoring of polynomials whose degree is greater that two, hence cubics—I then tell my students that although we have a formula for quadratics, one also exists for cubics, but is no longer taught. Basically we use the rational root theorem and long division to break the polynomial into more manageable pieces. Linear factors are certainly the goal, but we feed students polynomials that routinely reduce to very *nice* linear factors. In all, our examples where the degree is greater than two would always have enough rational roots to reduce the

[2]Rutgers University was my first chance at teaching.

polynomial to at most degree two. The degree two polynomials don't necessarily have to have *nice* roots, because they have an easily derivable quadratic formula to massage those *nasty ones* out.

So one day, in my introductory calculus class, I inadvertently was presented a cubic that did not have rational roots. Here I asked the students to find the real root of a cubic mainly because we needed the real root in order to analyze the derivative. Here goes, we found

$$f'(x) = x^3 + 6x - 2,$$

so we then proceeded to look for rational roots. Of course we knew what to test, but much to my chagrin, we were unable to find one. So, I took it as a teaching opportunity and told my students the story about my teacher—the PhD from Columbia University that lamented the loss of solutions to cubics in the curriculum—and then told them to solve the problem using Mathematica, or whatever technique worked for them. Here's what some[3] gave back to me,

$$x \approx 0.327480002073,$$

which was really more than *good* enough for what we were doing. They obtained this answer on a calculator, either by graphing the function and looking for the roots, or by using the calculator's solve function. However others went on to find exact answers[4] and reported back to me

$$x = -\sqrt[3]{2} + \sqrt[3]{4}, \qquad x = -\frac{1 - i\sqrt{3}}{\sqrt[3]{2}} + \frac{1 + i\sqrt{3}}{\sqrt[3]{4}}, \qquad x = \frac{1 - i\sqrt{3}}{\sqrt[3]{4}} - \frac{1 + i\sqrt{3}}{\sqrt[3]{2}},$$

after feeding Mathematica the equation. The solutions that involved *imaginary* numbers were summarily dismissed as being unnecessary because we're only working with real numbers—as our calculus curriculum demands.

Another student—not an overly bright or motivated one for that matter—did an Internet search on solving cubics and came across Cardano's[5] method. I was so impressed that I published a *doctored* picture of Einstein (Figure 1) facetiously complementing[6] the student, many of the students were besmirched because this particular student was not the brightest amongst them, but he did what others did not: he investigated another option, a difficult one at that. However, I again was reminded about my teacher who lamented our inability to solve cubics, and that now might be the time to make sense of it all. Not the recipe, but the actual discovery. Although this is not a rigorous historical treatment of what Cardano *et. al.* did, I think it's at least a possible path, albeit one that I can follow as being reasonable, and easy to follow for my students that express interest.

I can't help but think that the author of the original calculus problem had an ulterior motive. Especially since the root of the derivative was clearly of the form[7]

$$x^3 = 3px + 2q,$$

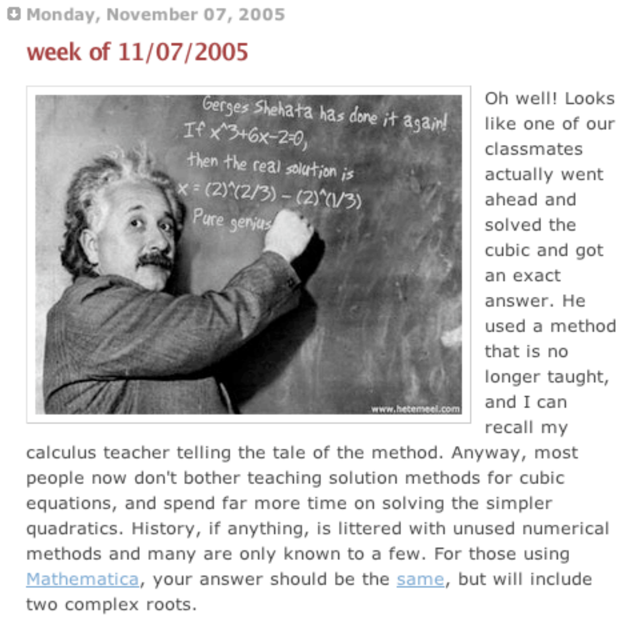
[3]Not all students solved the problem, and those that do will often share their results with those who cannot, or will not do the work.

[4]A peculiar bias I have that's mainly a result of my mathematics teachers insistence on not using them.

[5]Gerolamo Cardano (1501-1576) is credited with this method, but it is really a result of a hint provided by Niccolò Tartaglia, but I'm sure that path extends *ad infinitum*. Interested students should be referred to Stillwell's *Mathematics and Its History* for a more accurate history.

[6]It reads, "Gerges Shehata has done it again! If $x^3 + 6x - 2 = 0$, then the real solution is $x = \sqrt[3]{4} - \sqrt[3]{2}$ . . . Pure genius!"

[7]Often referred to as the *depressed* cubic, or del Farro's cubic equation.


Monday, November 07, 2005

week of 11/07/2005

Oh well! Looks like one of our classmates actually went ahead and solved the cubic and got an exact answer. He used a method that is no longer taught, and I can recall my calculus teacher telling the tale of the method. Anyway, most people now don't bother teaching solution methods for cubic equations, and spend far more time on solving the simpler quadratics. History, if anything, is littered with unused numerical methods and many are only known to a few. For those using Mathematica, your answer should be the same, but will include two complex roots.

Figure 1: Detail from my class web page.

with $p = -2$ and $q = 1$, and the solution is visually represented by the intersection of a line ($y = 3px + 2q$) and a simple cubic ($y = x^3$). In Cardano's *Ars Magna*,[8] it was shown to have a solution

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}},$$

and this will be clearly shown in the next section. This is exactly how my student found his exact real answer, he just plugged the p and q in as follows:

$$x = \sqrt[3]{1 + \sqrt{1+8}} + \sqrt[3]{1 - \sqrt{1+8}} = \sqrt[3]{4} - \sqrt[3]{2}.$$

All is well!

Maybe not, because there's something inherently wrong here, a fact that this particular example escaped. First it is clear to most pre-calculus students that a simple cubic and any line must intersect[9] at a real number, but what if $q^2 < p^3$, then we'll have a very strange answer indeed! The answer must be real—the point(s) can be clearly seen by all[10]—but what in God's name is this number?

4 Bombelli's Choice

As an example let's take Bombelli's[11] paradoxical choice

$$x^3 = 15x + 4,$$

[8] In Tristan Needham's *Visual Complex Analysis* it is stressed that this is the birthplace of complex numbers. This story has been told by many others, but I am mainly relying on Needham's version to retell this tale as if I were telling my students.

[9] At least once, but at most three times.

[10] This is not true of the quadratics, which is a more often used example of why complex numbers are needed.

[11] Rafael Bombelli (1526-1572) is credited with making sense of Cardano's formula.

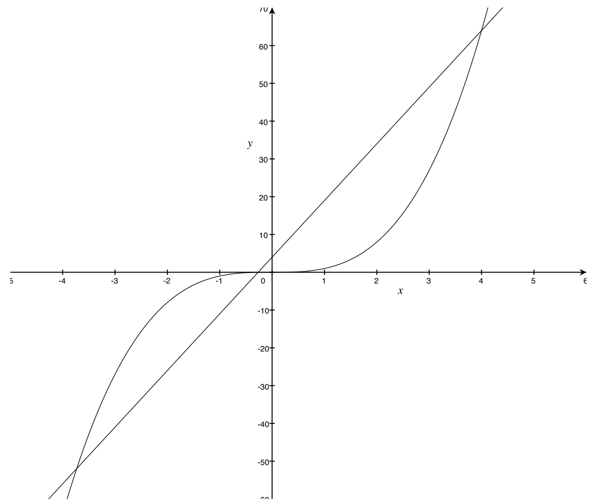


Figure 2: The graph of $y = x^3$ and $y = 15x + 4$

which has one very simple answer, that can be obtained by inspection, namely that $x = 4$. With today's calculators we could actually just take a look at graph and quickly glean from the graph that we have three real roots, with the 4 clearly hitting us, but the other two may not be so clear, but we could easily find the other two as follows,

$$\frac{x^3 - 15x - 4}{x - 4} = x^2 + 4x + 1.$$

Now, using the quadratic formula on the quotient, we have

$$x^2 + 4x + 1 = 0 \quad \Rightarrow \quad x = -2 \pm \sqrt{3}.$$

However, using Cardano's formula yields an oddity, in Bombelli's paradoxical choice where $p = 5$ and $q = 2$. Again just plugging it in yields[12]

$$x = \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

All is not well! And Bombelli clearly knew this, and he set out to torture this beast to produce 4, that is to show,

$$4 = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

Where he'd eventually let

$$\sqrt[3]{2 + 11i} = 2 + i \quad \text{and} \quad \sqrt[3]{2 - 11i} = 2 - i,$$

just simply adding those together would, as expected, yield 4, that of course means that cubing $2 \pm i$ should yield $2 \pm 11i$, and it does.

$$(2 \pm i)^3 = 2 \pm 11i$$

[12]This notation is modern, and was not used by Bombelli.

After observing this little *trick* (I did it myself[13]) I bet a really motivated student could find these as well,

$$\sqrt[3]{2+11i} = -1 - \frac{\sqrt{3}}{2} + \frac{\sqrt{13-4\sqrt{3}}}{2} i \quad \text{and} \quad \sqrt[3]{2-11i} = -1 - \frac{\sqrt{3}}{2} - \frac{\sqrt{13-4\sqrt{3}}}{2} i,$$

finally,

$$\sqrt[3]{2+11i} = -1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{13+4\sqrt{3}}}{2} i \quad \text{and} \quad \sqrt[3]{2-11i} = -1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{13+4\sqrt{3}}}{2} i.$$

Thus we have a perfectly *real* problem that is demanding us to use complex numbers to solve it. Certainly a mathematical epiphany for Bombelli, and really makes more sense as to why we need complex numbers to solve problems. However, in my experience it seems that the the complex story started with quadratics, but that just doesn't make any sense after reading about Bombelli's work.

From Nahin's book,[14] he writes:

> When the imaginary number $\sqrt{-1}$ is first introduced to high school students it is common to read something like the following (which, actually, I've taken from a college textbook[15]): "The real equation $x^2 + 1 = 0$ led to the invention of i (and also $-i$) in the first place. That was declared to be the solution and the case was closed." Well, this is simple to read and easy to remember but, as you now know, it is also not true. When the early mathematicians ran into $x^2 + 1 = 0$ and other such quadratics they simply shut their eyes and called them "impossible." They certainly did not invent a solution for them. The breakthrough for $\sqrt{-1}$ came not from quadratic equations, but rather from cubics which clearly had real solutions but for which the Cardan formula produced formal answers with imaginary components.

Examples abound and they must have been perplexing, especially in Bombelli's example where it is clear that three solutions exists. What is most impressive here is that this very strange expression,

$$\sqrt[3]{2+11i} + \sqrt[3]{2-11i},$$

actually produces three (easily seen and verified) real numbers,

$$\left\{-2-\sqrt{3},\ -2+\sqrt{3},\ 4\right\}.$$

5 Here Goes . . .

The most general third degree polynomial will be of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

[13] Looking for $a \pm bi$, we know what the a is, so we just need to solve for the b.
[14] *An Imaginary Tale.*
[15] Gilbert Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge Press 1986, p. 330.

where $a_3 \neq 0$. Since we're just looking for the roots, we can rewrite $f(x) = 0$, and then divide through by a_3, as follows:

$$\begin{aligned} 0 &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ 0 &= \frac{a_0}{a_3} + \frac{a_1}{a_3} x + \frac{a_2}{a_3} x^2 + \frac{a_3}{a_3} x^3 \end{aligned}$$

To simplify the coefficients let

$$\frac{a_0}{a_3} = C, \quad \frac{a_1}{a_3} = B, \quad \text{and} \quad \frac{a_2}{a_3} = A,$$

giving a form

$$0 = x^3 + Ax^2 + Bx + C.$$

Certainly, the roots of $f(x)$ will be the same roots as $x^3 + Ax^2 + Bx + C$. So let

$$g(x) = x^3 + Ax^2 + Bx + C.$$

The second derivative is a linear function ($g''(x) = 6x + 2A$) giving a a clear point of infection at $x = -A/3$ on the graph of $g(x)$. Using standard translation, I'd like to pull this point of inflection $A/3$ units to the right, that is

$$g\left(x - \frac{A}{3}\right) = \left(x - \frac{A}{3}\right)^3 + A\left(x - \frac{A}{3}\right)^2 + B\left(x - \frac{A}{3}\right) + C$$

Which, when expanded will yield

$$g\left(x - \frac{A}{3}\right) = x^3 + \left(\frac{3B - A^2}{3}\right) x + \left(\frac{2A^3 - 9AB}{27}\right).$$

As before, we need to simplify the coefficients, as follows

$$\frac{3B - A^2}{3} = b, \quad \text{and} \quad \frac{2A^3 - 9AB}{27} = c,$$

giving a form

$$0 = x^3 + bx + c.$$

This, of course, wasn't by accident, and one just needs to look at what Cardano knew, especially the work of del Ferro and Tartaglia. Here, the important point is that we can rewrite any third degree polynomial into this form, and this is where I believe Cardano's genius started.

6 Solving $x^3 = 3px + 2q$

God only knows how much effort goes into solving problems that may seem pointless to almost everyone. I've been accused of giving my students extra credit problems that require a level of *pointless* perseverance that is rarely expected by other teachers. I mainly do this to encourage my students to think, become frustrated, and to think again and again. The hope is that eventually they will stumble upon the answer, in Edison's words "I am not discouraged, because every wrong attempt discarded is another step forward." As a teacher in an intellectually poor urban environment, I know all too well that many don't even try because each step taken results in incresing frustration. I cannot help but praise Edison for seeing that effort, in itself, is sufficient reason to proceed forward and is really independent of our eventual success. Also, success can only be really enjoyed if we put effort into what we do—that is, we need to work at it. In this light, Cardano must have discarded an awful lot of work (pointless effort?) before stumbling upon some form of the following work.[16]

Using $x = s + t$ and then substituting into $x^3 = 3px + 2q$, yields

$$\begin{aligned} x^3 &= 3px + 2q \\ (s+t)^3 &= 3p(s+t) + 2q \\ s^3 + 3s^2t + 3st^2 + t^3 &= 3ps + 3pt + 2q \end{aligned}$$

Now if $st = p$ and $s^3 + t^3 = 2q$ we have

$$\begin{aligned} s^3 + 3s^2t + 3st^2 + t^3 &= 3ps + 3pt + 2q \\ \left(s^3 + t^3\right) + 3s(st) + 3(st)t + &= 3ps + 3pt + 2q \\ 2q + 3ps + 3pt &= 3ps + 3pt + 2q \end{aligned}$$

Now eliminate t between $st = p$ and $s^3 + t^3 = 2q$, by letting $t = p/s$ from the first equation, then substituting it into the second.

$$s^3 + t^3 = 2q \quad \Rightarrow \quad s^3 + \left(\frac{p}{s}\right)^3 = 2q \quad \Rightarrow \quad s^6 + p^3 = 2s^3q \quad \Rightarrow \quad s^6 - 2s^3q + p^3 = 0$$

Now, using the quadratic formula to solve

$$s^6 - 2s^3q + p^3 = 0 \quad \Rightarrow \quad \left(s^3\right)^2 - 2q\left(s^3\right) + p^3 = 0,$$

where

$$s^3 = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = q \pm \sqrt{q^2 - p^3}$$

So we have

$$s = \sqrt[3]{q - \sqrt{q^2 - p^3}} \quad \text{or} \quad s = \sqrt[3]{q + \sqrt{q^2 - p^3}}.$$

We're getting close! Now use these solutions in $s^3 + t^3 = 2q$ to solve for t^3, as follows

$$s^3 + t^3 = 2q \quad \Rightarrow \quad \left(q \pm \sqrt{q^2 - p^3}\right) + t^3 = 2q \quad t^3 = q \pm \sqrt{q^2 - p^3}$$

So we know have

$$t = \sqrt[3]{q - \sqrt{q^2 - p^3}} \quad \text{or} \quad t = \sqrt[3]{q + \sqrt{q^2 - p^3}},$$

however, we must have $s^3 + t^3 = 2q$, and the only way for this to occur is if we take

$$s^3 = q - \sqrt{q^2 - p^3} \quad \text{and} \quad t^3 = q + \sqrt{q^2 - p^3}$$

or, equivalently, we could also get

$$s^3 = q + \sqrt{q^2 - p^3} \quad \text{and} \quad t^3 = q - \sqrt{q^2 - p^3}.$$

So, no matter how we do it, we'll finally have

$$x = s + t = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

This, of course, is exactly what we were aiming for.

[16]I'm really not sure how Cardano proceeded, and I've seen variations, but I think this is a reasonably approach which is outlined in the exercises (page 45) of Needham's *Visual Complex Analysis*.

7 Closing Statement

As a teacher, I believe we are routinely confronted by students who have no desire to wade trough the tedium of mathematical manipulation. I know, from experience, that the work presented in this paper would kill off 99% of my students, that is to be understood. The point of this paper being, is that I don't lament that Cardano's method has been lost in our educational structure, but that we've lost the surprise that Bombelli experienced, and math teachers have succumbed to becoming revisionist historians by presenting quadratics as the root cause for extending the real number system.

If I were presenting the techniques presented in this short paper to students, I would also emphasize using them. So in that light, I have prepared the following examples that some students should be able to do at this point. However, for practical purposes, I'd strongly suggest that these techniques only be employed if a rational root does not exists. Again, the main point of this paper was to emphasize Bombelli's observation and acceptance that

$$\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$$

isn't so strange after all. And I think it's safe to say that complex numbers are still being ignored today in our curriculum, but that the practical fields of physics and electrical engineering have embraced them because they make their work easier.

1. Find one root using the technique described in this paper. The numbers will be difficult, and a calculator will help greatly.

$$f(x)=(2x-1)(3x+2)(5x-4)=30x^3-19x^2-14x+8.$$

Work: Clearly we know the roots are

$$x=\frac{1}{2}\quad x=-\frac{2}{3}\quad x=\frac{4}{5},$$

but we need to use the techniques used in this paper. So, set $f(x)=0$ and divide both sides of the equation by 30.

$$\begin{aligned} 0 &= 30x^3-19x^2-14x+8 \\ 0 &= x^3-\frac{19}{30}x^2-\frac{7}{15}x+\frac{4}{15} \end{aligned}$$

Certainly both of these equations have the same solution. Now let

$$g(x)=x^3-\frac{19}{30}x^2-\frac{7}{15}x+\frac{4}{15},$$

and find the x-coordinate of inflection point by setting $g''(x)=0$.

$$g''(x)=6x-\frac{19}{15}=0\quad\Rightarrow\quad x=\frac{19}{90}$$

Now find

$$\begin{aligned} g\left(x+\frac{19}{90}\right) &= \left(x+\frac{19}{90}\right)^3-\frac{19}{30}\left(x+\frac{19}{90}\right)^2-\frac{7}{15}\left(x+\frac{19}{90}\right)+\frac{4}{15} \\ &= x^3-\frac{1621}{2700}x+\frac{54431}{364500} \end{aligned}$$

Now rewrite this equation in the form $x^3 = 3px + 2q$.

$$\begin{aligned} x^3 - \frac{1621}{2700}x + \frac{54431}{364500} &= 0 \\ x^3 &= \frac{1621}{2700}x - \frac{54431}{364500} \\ x^3 &= 3 \cdot \frac{1621}{8100}x + 2 \cdot \left(-\frac{54431}{729000}\right) \end{aligned}$$

Using the values of p and q in the following formula yields three answers, but just considering the principle roots, we have[17]

$$\begin{aligned} x &= \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} \\ &= \sqrt[3]{-\frac{54431}{729000} + \frac{77\sqrt{3}}{2700}i} + \sqrt[3]{-\frac{54431}{729000} - \frac{77\sqrt{3}}{2700}i} \\ &= \frac{53}{90} \end{aligned}$$

Now we finally have at least one answer

$$\frac{53}{90} + \frac{19}{90} = \boxed{\frac{4}{5}},$$

resulting in a factor of $(5x - 4)$, which can then be used as a divisor of $f(x)$ yielding a factorable quotient $6x^2 + x - 2 = (2x - 1)(3x + 2)$, thus the other two expected roots, $\{-2/3,\ 1/2\}$.

2. Perform the indicated long division.

$$\frac{x^3 - (1621/2700)\,x + 54431/364500}{x - 53/90}$$

Work: Clearly tedious work!

$$\frac{x^3 - (1621/2700)\,x + 54431/364500}{x - 53/90} = \boxed{x^2 + \frac{2385}{4050}x - \frac{1027}{4050}}$$

3. Find the roots of the quotient in the prior example by using the quadratic formula.

Work: Clearly tedious work again!

$$0 = x^2 + \frac{2385}{4050}x - \frac{1027}{4050} \quad \Rightarrow \quad \boxed{x = \frac{13}{45} \quad \text{or} \quad x = -\frac{79}{90}}$$

[17] My calculator actually yielded $0.5\overline{88}$ which is easily rewritten.

4. Use the prior example to find the remaining[18] roots of

$$f(x) = (2x-1)(3x+2)(5x-4) = 30x^3 - 19x^2 - 14x + 8.$$

Work: If you got this far, this should be easy!

$$x = \frac{13}{45} + \frac{19}{90} = \boxed{\frac{1}{2}} \quad \text{or} \quad x = -\frac{79}{90} + \frac{19}{90} = \boxed{-\frac{2}{3}}$$

5. Find the three roots of

$$\sqrt[3]{\sqrt{3}+i}$$

Work: This method is taught in pre-calculus.

$$z_k = \sqrt[3]{\sqrt{3}+i} = \sqrt[3]{2} \cdot \left[\cos\left(\frac{\pi/3 + 2\pi k}{3}\right) + i\sin\left(\frac{\pi/3 + 2\pi k}{3}\right)\right], \quad k = 0, 1, 2.$$

If $k = 0$,

$$z_0 = \sqrt[3]{2} \cdot \left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right];$$

if $k = 1$,

$$z_1 = \sqrt[3]{2} \cdot \left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right];$$

and if $k = 2$,

$$z_1 = \sqrt[3]{2} \cdot \left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right].$$

References

[1] Spiegel, Murray R. (1999). *Complex Variables, With an Introduction to Conformal Mapping and Its Application*. Schaum's Outline Series, McGraw-Hill, New York, NY.

[2] Boas, Ralph P. (1987). *Invitation to Complex Analysis, First Edition*. Random House Inc., New York, NY.

[3] Needham, Tristan (2005). *Visual Complex Analysis*. Clarendon Press, Oxford, UK.

[4] Nahin, Paul J. (1998). *An Imaginary Tale*. Princeton University Press, Princeton, NJ.

[18] We already have root 4/5.