Exercise Three: Proclamations and Proofs

Theorem 1. *The product of any two even integers is an even integer.*

Proof. Suppose m and n are even integers.

By the definition of even, $m = 2j$ and $n = 2k$ where j and k are integers.

Therefore, $mn = (2j)(2k) = 2(2jk)$ by basic properties of multiplication.

Since the integers are closed under multiplication, jk is also an integer.

And this same property of closure also tells us that $2jk$ is an integer.

So, mn is twice another integer $(2jk)$, meaning mn is even, as desired. □

Definition. An integer k is *odd* if $k = 2j + 1$ where j is an integer.

Theorem 2. *The sum of any two odd integers is an even integer.*

Proof. Suppose m and n are odd integers.

By the definition of odd, $m = 2j + 1$ and $n = 2k + 1$ where j and k are integers.

Therefore, $m + n = (2j + 1) + (2k + 1) = 2j + 2k + 2 = 2[(j + k) + 1])$ by basic properties of addition and multiplication.

Since the integers are closed under addition, $j + k$ is also an integer.

And this same property of closure also tells us that $(j + k) + 1$ is an integer.

So, $m + n$ is twice another integer $[(j + k) + 1]$, meaning $m + n$ is even, as desired. □

Theorem 3. *Every even integer greater than two is the sum of two primes.*

This is the famous Goldbach Conjecture. As of this writing, it is unproven.