Exercise Four: Set Notation, Greek and Hebrew Letters

If $A = \{\heartsuit, \spadesuit\}$ and $B = \{\epsilon, \theta\}$, then
$A \cup B = \{\heartsuit, \spadesuit, \epsilon, \theta\}$
$A \cap B = \emptyset$
$\mathcal{P}(A \cap B) = \{\emptyset\}$

Theorem 1. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. We assume $A \subseteq B$ and $B \subseteq C$, and we want to show $A \subseteq C$.
Suppose $x \in A$. We must show that $x \in C$ also.
Since $A \subseteq B$, we know $x \in B$.
Similarly, since $B \subseteq C$, we also know $x \in C$.
But this means $A \subseteq C$, as desired. □

Theorem 2. *If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Proof. We assume $A \subseteq B$, and we want to show $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
Suppose $X \in \mathcal{P}(A)$. We must show $X \in \mathcal{P}(B)$ also.
By the definition of power set, $X \in \mathcal{P}(A)$ means that X is a subset of A.
Since $X \subseteq A$ and $A \subseteq B$, the previous theorem tells us that $X \subseteq B$ also.
Then, using the definition of power set again, this means $X \in \mathcal{P}(B)$.
Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, as desired. □

Puzzling Power Sets

$\mathcal{P}(\emptyset) = \{\emptyset\}$
$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}\}$