

1 Two Theorems and One Principle

Introductory Question: If a biker travels 45 kilometers in 3 hours, can we guarantee that her speed must have read 15 kph at least once during the trip? Can we guarantee it will read 12 kph at least once during the trip? How about 18 kph?

Introductory Answer: I think it only stands to reason that her speedometer must have read 15 kph at least once during the trip. however we can not guarantee any other speed. I want to point out, that there's a lot of mathematics that *stands to reason*, but mathematics must actually pass a more rigorous test and that is where proof comes into play.

Rolle's Theorem: Let f be a function that satisfies the following three hypothesis:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem: Let f be a function that satisfies the following two hypothesis:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem Example: Verify that $f(x) = x^3 - 3x^2 + 2x + 5$, on the interval $[0, 2]$ satisfies all three conditions and then find the value for c .

Work: First of all, $f(x) = x^3 - 3x^2 + 2x + 5$ is continuous everywhere,² so it is continuous on the interval $[0, 2]$. Polynomials are differentiable everywhere, and its derivative is $f'(x) = 3x^2 - 6x + 2$. Evaluating f at its endpoints we have $f(0) = f(2) = 5$. To find the $c \in (0, 2)$ we need to solve

$$\begin{aligned} f'(x) &= 0 \\ 0 &= 3x^2 - 6x + 2 \\ x &= \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{3 \pm \sqrt{3}}{3} \end{aligned}$$

Both answers are on the interval, so we found two values for c . Here's a graph.

¹This document was prepared by Ron Bannon (ron.bannon@gmail.com) using L^AT_EX 2_ε.

²It's a polynomial.

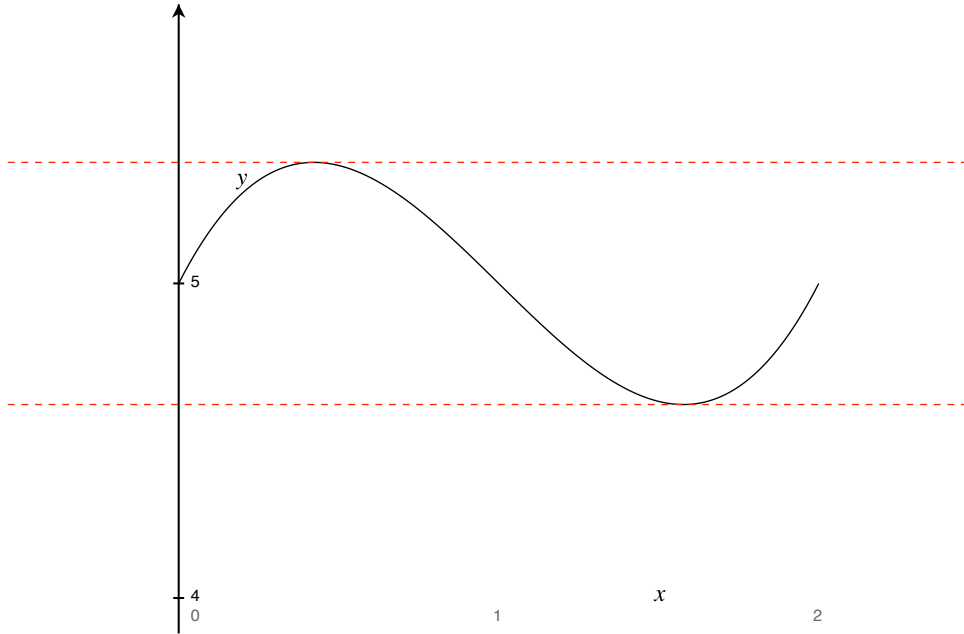


Figure 1: The graph of $f(x)$ and the two tangent lines at $x = \frac{3 \pm \sqrt{3}}{3}$.

Mean Value Theorem Example: Verify that $f(x) = \frac{x}{x+2}$, on the interval $[1, 4]$ satisfies all two conditions and then find the value for c .

Work: First of all, $f(x)$ is continuous on its domain which includes the interval $[1, 4]$.³ Its derivative is

$$f'(x) = \frac{2}{(x+2)^2}.$$

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Making the substitutions and solving for c , we have.

$$\begin{aligned} \frac{2}{(c+2)^2} &= \frac{f(4) - f(1)}{4 - 1} \\ \frac{2}{(c+2)^2} &= \frac{1}{9} \\ 18 &= (c+2)^2 \\ \pm 3\sqrt{2} &= c+2 \\ \pm 3\sqrt{2} - 2 &= c \end{aligned}$$

However we only have one value for c that's in the interval, that is $c = 3\sqrt{2} - 2$. Here's a graph.

³It's a rational function whose domain is all real numbers except 2 and is differentiable everywhere in its domain.

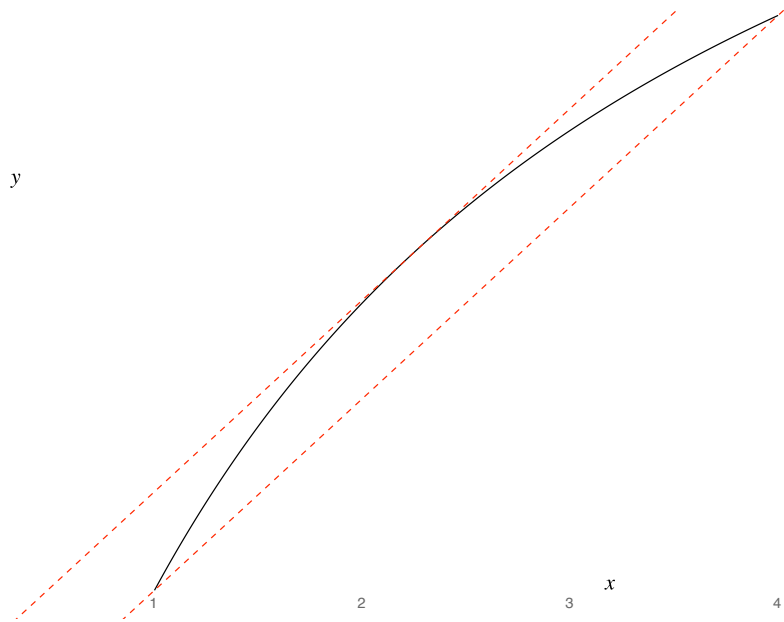


Figure 2: The graph of $f(x)$ and the tangent line at $x = 3\sqrt{2} - 2$.

It should be further noted that the line drawn between the endpoints is parallel to the tangent line. You should be clear about which one of these two lines is tangent to f .

1. Show using the Intermediate Value Theorem and Rolle's Theorem that

$$f(x) = x^7 + 4x^5 + 3x + 5$$

has exactly one real root and that root is between -1 and 0 .

Work: Since $f(x)$ is continuous everywhere and $f(-1) = -3 < 0$ and $f(0) = 5 > 0$ we know that by the Intermediate Value Theorem that a root exists between -1 and 0 , that is, there exists at least one $c \in (-1, 0)$ such that $f(c) = 0$.

Now, the derivative of $f(x)$ is $f'(x) = 7x^6 + 20x^4 + 3$ is always positive. So now assume that two roots exist, $f(a) = f(b) = 0$, where $a < b$, since the polynomial is continuous on $[a, b]$ and differentiable on (a, b) , Rolle's Theorem implies that there is a number $r \in (a, b)$ such that $f'(r) = 0$, but that's not possible because $f'(x) = 7x^6 + 20x^4 + 3$ is always positive. This contradiction shows that $f(x)$ cannot have two real roots. Since we've already shown the existence of a root, the conclusion is that there's only one real root. Here's the graph.

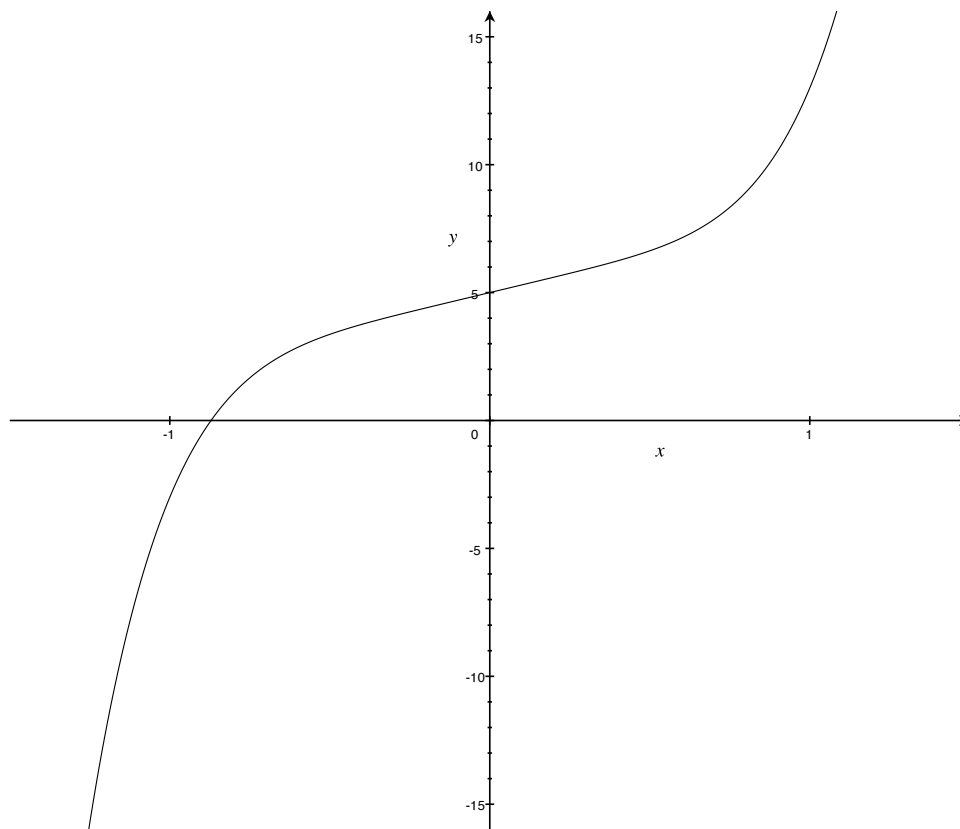


Figure 3: Partial graph of $f(x) = x^7 + 4x^5 + 3x + 5$.

The Racetrack Principle: Suppose that $g(x)$ and $h(x)$ are continuous on $[a, b]$ and differentiable on (a, b) , and that $g'(x) \leq h'(x)$ for all $x \in (a, b)$.

(a) If $g(a) = h(a)$, then $g(x) \leq h(x)$ for all $x \in [a, b]$.

(b) If $g(b) = h(b)$, then $g(x) \geq h(x)$ for all $x \in [a, b]$.

I think you might want to think about these two functions as being in a race. Similar to two people running a marathon where there's a beginning and an end. So let $g(x)$ and $h(x)$ be two runners, where $g(x)$ is slower than $h(x)$. Considering that they start together, then $g(x)$ will be behind $h(x)$ except at the moment they start; if $g(x)$ is given a head start and wins, then he always had to be in front.

The Racetrack Principle Example: Show that $\sin x \leq x$ for all $x \geq 0$.

Work: Let $g(x) = \sin x$ and $h(x) = x$, which are both continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, and that $g'(x) = \cos x \leq 1 = h'(x)$ for all $x \in (0, \infty)$.

(a) If $g(0) = h(0) = 0$, then $g(x) \leq h(x)$ for all $x \in [0, \infty)$.

Q.E.D.

Assignment:

1. Read §4.2.
2. You should be able to do the following problems.

4.2 The Mean Value Theorem: 1, 5, 8, 10, 11, 18, 21, 23, 28, 29, 34, 35.

WebAssign problems, similar to the ones above, will be posted and you need to get started

3. Again, don't fall behind!