

1 Important Features of f

Ah, here we are, once again, back in MTH-119 trying to graph a polynomial function.

$$f(x) = x^3 - 9x^2 - 4x + 36$$

As I hope you recall from your MTH-119 days, you need to factor² this polynomial, determine the x -intercepts, y -intercept, and do some very simple sign-analysis. After doing all this you were expected to make a pre-calculus sketch. You should also recall that your MTH-119 teacher most likely said that this was a rather crude method, and that you'd have to wait for calculus for better detail.

Let's start the MTH-119 process.

$$\begin{aligned} f(x) &= x^3 - 9x^2 - 4x + 36 \\ &= x^2(x - 9) - 4(x - 9) \\ &= (x - 9)(x^2 - 4) \\ &= (x - 9)(x - 2)(x + 2) \end{aligned}$$

Clearly the x -intercepts are: $(-2, 0)$, $(2, 0)$, and $(9, 0)$. And the y -intercept is: $(0, 36)$. Plotting these points and then doing the sign-analysis results in the following graph, albeit not as refined as my computer's version.

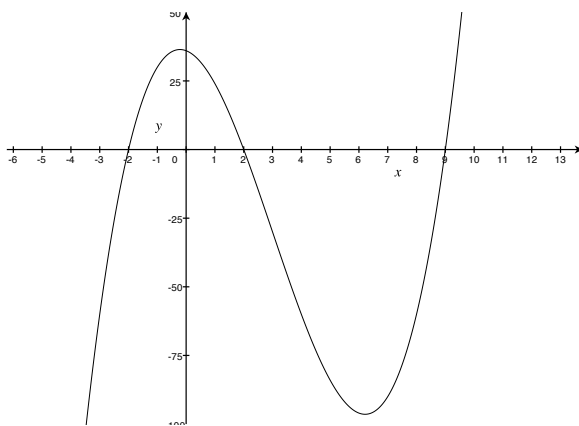


Figure 1: $f(x) = x^3 - 9x^2 - 4x + 36$

Now we will start the process of using the derivatives (calculus) to refine our understanding of f . Here are what the derivatives tell us about the shape of a curve?

1. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.

¹This document was prepared by Ron Bannon (ron.bannon@gmail.com) using L^AT_EX 2_ε.

²Rational Root Theorem, long division, *etc.*.

2. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.
3. If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.
4. If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

So let's take a look at $f(x)$, $f'(x)$, and $f''(x)$ on the same graph.

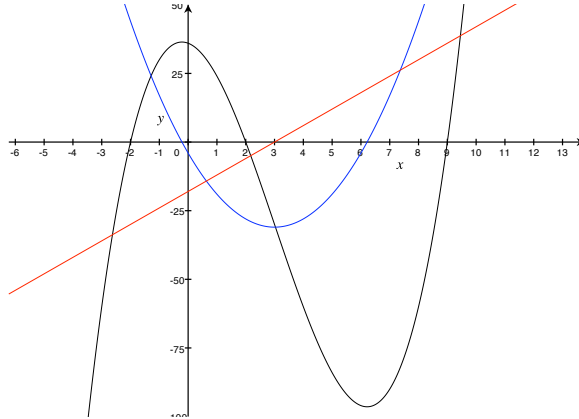


Figure 2: $f(x)$, $f'(x)$ in blue, and $f''(x)$ in red.

Time to think!

1. Find $f'(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

2. Find $f''(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

3. Find the local extrema on $f(x)$. What happened to f' at this point?

4. Find the point on $f(x)$ where the concavity changes. What happened to f'' at this point?

Time to check your thinking!

1. Find $f'(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

Work:

$$\begin{aligned}f(x) &= x^3 - 9x^2 - 4x + 36 \\f'(x) &= \boxed{3x^2 - 18x - 4}\end{aligned}$$

As I hope you recall from MTH-119, the necessary step is to find where $f'(x) = 0$ and then to use simple sign-analysis to determine when $f'(x) < 0$ and when $f'(x) > 0$.

$$3x^2 - 18x - 4 = 0 \quad \Rightarrow \quad x = \frac{18 \pm \sqrt{18^2 + 4 \cdot 3 \cdot 4}}{6} = \frac{9 \pm \sqrt{93}}{3}$$

You might want to make *sense* out of this irrational number by approximating³ it with a rational number. But I am interested in exact answers! So here's the interval where $f'(x) < 0$:

$$\boxed{\left(\frac{9 - \sqrt{93}}{3}, \frac{9 + \sqrt{93}}{3} \right)},$$

and here's the intervals where $f'(x) > 0$:

$$\boxed{\left(-\infty, \frac{9 - \sqrt{93}}{3} \right) \cup \left(\frac{9 + \sqrt{93}}{3}, \infty \right)}.$$

2. Find $f''(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

Work:

$$\begin{aligned}f'(x) &= 3x^2 - 18x - 4 \\f''(x) &= \boxed{6x - 18}\end{aligned}$$

It is getting easier, I hope. The interval where $f''(x) > 0$:

$$\boxed{(3, \infty)},$$

³-0.215 and 6.215

and the interval where $f''(x) < 0$:

$$\boxed{(-\infty, 3]}.$$

3. Find the local extrema on $f(x)$.

Work: The local extrema occurs on f where the first derivative is zero. The local maximum is:

$$\boxed{\left(\frac{9 - \sqrt{93}}{3}, f\left(\frac{9 - \sqrt{93}}{3}\right)\right)},$$

and the local minimum is

$$\boxed{\left(\frac{9 + \sqrt{93}}{3}, f\left(\frac{9 + \sqrt{93}}{3}\right)\right)}.$$

4. Find the point on $f(x)$ where the concavity changes.

Work: This point is called an inflection point.

$$\boxed{(3, f(3))}.$$

1.1 Definitions and Theorems

1. **Definition:** A function f has a *absolute maximum* (or *global maximum*) at c if $f(c) \geq f(x)$ for all x in f 's domain. The number $f(c)$ is called the *maximum value* of f on f 's domain. Similarly, f has a *absolute minimum* (or *global minimum*) at c if $f(c) \leq f(x)$ for all x in f 's domain. The number $f(c)$ is called the *minimum value* of f on f 's domain. The minimum and maximum values of f are called *extreme values* of f .
2. **Definition:** A function f has a *local maximum* (or *relative maximum*) at c if $f(c) \geq f(x)$ when x is near c .⁴ The number $f(c)$ is called the *local maximum value* of f . Similarly, f has a *local minimum* (or *relative minimum*) at c if $f(c) \leq f(x)$ when x is near c .⁵ The number $f(c)$ is called the *local minimum value* of f . The local minimum and local maximum values of f are called *local extreme values* of f .
3. **Extreme Value Theorem:** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.
4. **Fermat's Theorem:** If f has a local maximum or minimum at c , and $f'(c)$ exists, then $f'(c) = 0$

⁴The c must be in some open interval containing c .

⁵Again, the c must be in some open interval containing c .

5. **Definition:** A *critical number* of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.
6. **Theorem:** If f has a local maximum or minimum at c , then c is a critical number of f .

1.2 Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in the open interval (a, b) .
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

An Example: Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 4]$.

Work: Although not necessary, here's a graph.

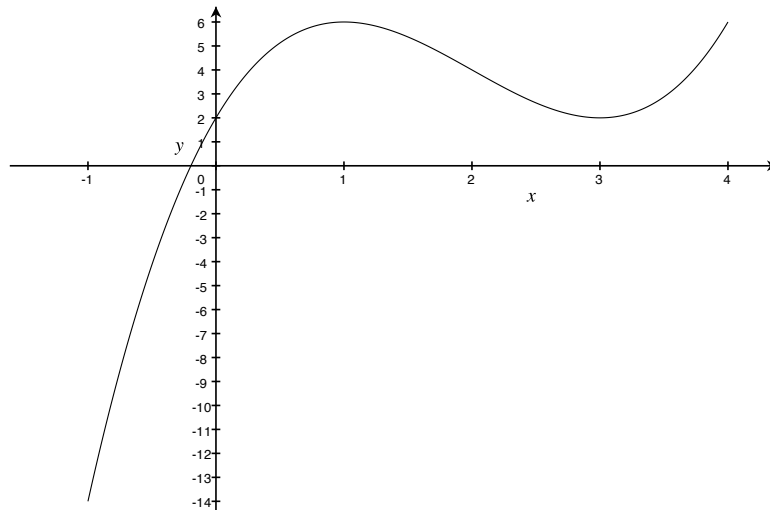


Figure 3: Complete graph of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1, 4]$.

We know that f is a polynomial and is continuous, and we're given a closed interval.

- $f'(x) = 3x^2 - 12x + 9 = 3(x - 3)(x - 1)$, the critical numbers are $x = 1$ and $x = 3$ and they are one the open interval $(-1, 4)$. $f(1) = 6$ and $f(3) = 2$.
- $f(-1) = -14$ and $f(4) = 6$.
- The absolute maximum value of f is 6; the absolute minimum value of f is -14 .

1.3 Let's Summarize

What Derivatives Tell Us About a Function and its Graph

- If $f' > 0$ on an interval, then f is *increasing* on that interval.
- If $f' < 0$ on an interval, then f is *decreasing* on that interval.
- If $f'' > 0$ on an interval, then f is *concave up* on that interval.
- If $f'' < 0$ on an interval, then f is *concave down* on that interval.

Local Maxima and Minima

Suppose a is a point in the domain of f

- f has a *local minimum* at a if $f(a)$ is less than or equal to values of f for points near a . Near a means that there needs to be points to the left and right of a .
- f has a *local maximum* at a if $f(a)$ is greater than or equal to values of f for points near a . Again, near a means that there needs to be points to the left and right of a .

Finding Local Extrema

For any function f , a point a in the domain of f where $f'(a) = 0$ or $f'(a)$ is undefined is called a *critical point* of the function. In addition, the point $(a, f(a))$ on the graph of f is also called a *critical point*.

The first-derivative test for local maxima and minima. Suppose a is a *critical point* of a continuous function f .

- If f' changes from negative to positive at a , then f has a local minimum at a .
- If f' changes from positive to negative at a , then f has a local maximum at a .

The second-derivative test for local maxima and minima. Suppose a is a *critical point* of a continuous function f .

- If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a .
- If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a .
- If $f'(a) = 0$ and $f''(a) = 0$, then the test tells us nothing.

Concavity and Points of Inflection

A point at which the graph of f changes concavity is called an *inflection point*.

1.4 Examples

The following examples will use the same methods that were taught in MTH-119, but now we are expected to refine our understanding of functions using calculus.

1. God is in the details! Or, how the *devil* is revealed in the calculus!

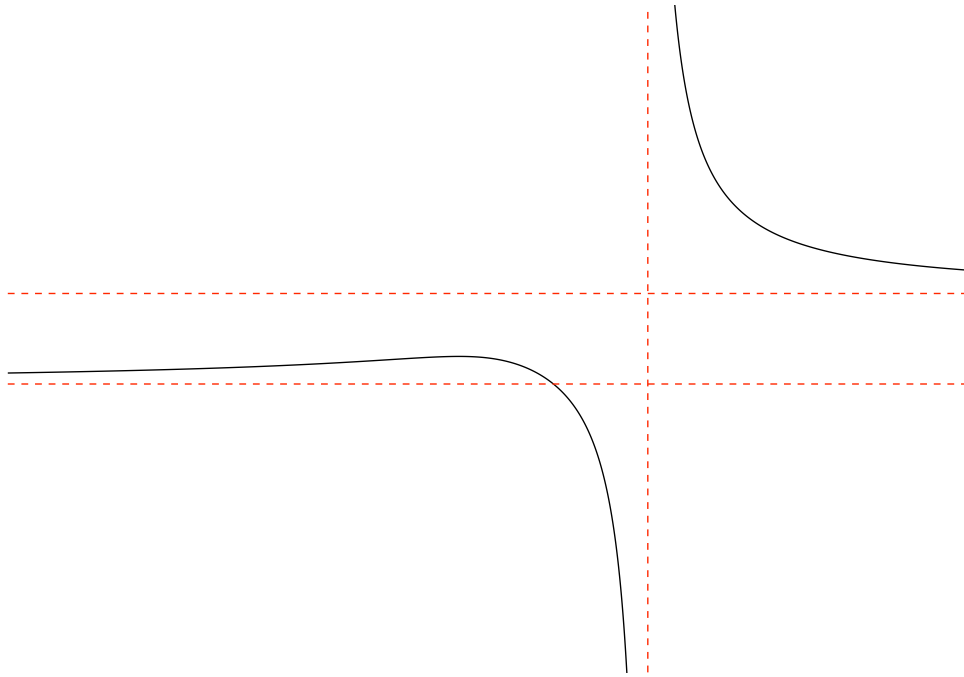


Figure 4: A lone graph of $f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$.

Yes, that's the graph of a function, properly scaled, but with no references to tag it down.

- (a) Are there any x -intercepts?

(b) Are there any y -intercepts?

(c) A vertical asymptote is indicated, what is its equation?

(d) Two horizontal asymptotes are indicated, what are their equations?

(e) Find the point(s) on f that are local extrema. But first verify that

$$f'(x) = -\frac{10x + 3}{(3x - 5)^2 \sqrt{2x^2 + 1}}.$$

(f) Find the interval(s) where f is increasing.

(g) Find the interval(s) where f is decreasing.

Answers follow.

- (a) Are there any x -intercepts?

Work: Well, $2x^2 + 1$ is never zero if we're restricted to using real numbers, so $f(x)$ can not be zero. That is, no x -intercepts.

- (b) Are there any y -intercepts?

Work: Just set $x = 0$, and you'll get $(0, -1/5)$.

- (c) A vertical asymptote is indicated, what is its equation?

Work: Easy, at least I think so. Just set $3x - 5$ to zero and solve for x . Of course you'll get $x = 5/3$. Remember that division by zero is a mortal sin and any sensible graph will avoid it like the plague.

- (d) Two horizontal asymptotes are indicated, what are their equations?

Work: Clearly they are different. We'll need to take limits as $x \rightarrow \pm\infty$.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} &= \frac{\sqrt{2}}{3} \\ \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} &= -\frac{\sqrt{2}}{3}\end{aligned}$$

So the two vertical asymptotes are:

$$\boxed{y = -\frac{\sqrt{2}}{3} \quad \text{and} \quad y = \frac{\sqrt{2}}{3}}.$$

- (e) Find the point(s) on f that are local extrema. But first verify that

$$f'(x) = -\frac{10x + 3}{(3x - 5)^2 \sqrt{2x^2 + 1}}.$$

Work: The only candidate for extrema is $x = -3/10$. The derivative on $(-\infty, -3/10)$ is always positive, and the derivative on $(-3/10, 5/3)$ is always negative. Certainly the graph, although not tagged down to an axis is indicating this, so we clearly have a local maximum at

$$\boxed{\left(-\frac{3}{10}, -\sqrt{\frac{2}{59}}\right)}$$

- (f) Find the interval(s) where f is increasing.

Work: The derivative on $(-\infty, -3/10)$ is always positive, that is where f is increasing.

(g) Find the interval(s) where f is decreasing.

Work: The derivative on $\boxed{(-3/10, 5/3) \cup (5/3, \infty)}$ is always negative, that is where f is decreasing.

2. You should be able to look at the following graph and discuss where the function is increasing, and decreasing; furthermore you should be able to determine where the derivative is positive, negative, undefined, and zero; better still, you should be able to determine where the function is concave up and concave down.

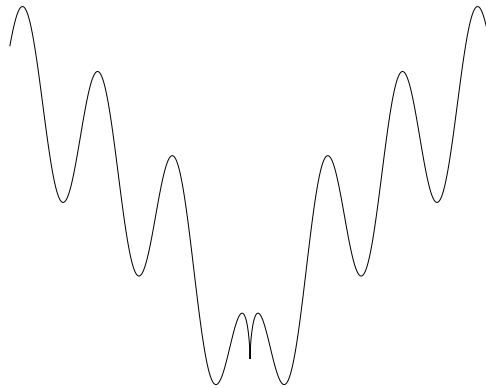


Figure 5: Partial graph of $f(x) = \sqrt{|x|} + \cos x$.

Now, let's look at a simpler example!

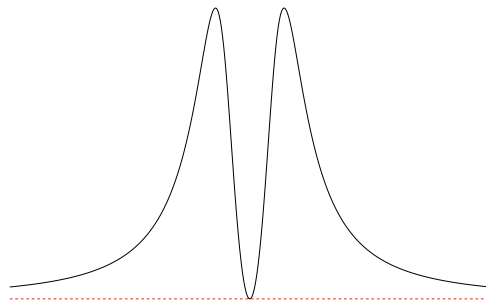


Figure 6: Partial graph of $f(x) = 2 + \frac{x^2}{x^4 + 1}$.

(a) Are there any x -intercepts?

(b) Are there any y -intercepts?

(c) A horizontal asymptote is indicated, what is its equation?

(d) Find the point(s) on f that are local extrema. But first verify that

$$f'(x) = -\frac{2x(x^4 - 1)}{(x^4 + 1)^2}.$$

(e) Find the interval(s) where f is increasing.

(f) Find the interval(s) where f is decreasing.

(g) Find the inflection point(s) on f . But first verify that

$$f''(x) = \frac{2(3x^8 - 12x^4 + 1)}{(x^4 + 1)^3}.$$

(h) Find the interval(s) where f is concave up.

(i) Find the interval(s) where f is concave down.

Answers follow.

(a) Are there any x -intercepts?

Work: The function can not equal zero when x is restricted to the real numbers. So we can safely say there are none.

(b) Are there any y -intercepts?

Work: Just set $x = 0$ and you'll get (0, 2).

(c) A horizontal asymptote is indicated, what is its equation?

Work: Clearly one is indicated by the red dashed line. We'll need to take limits as $x \rightarrow \pm\infty$.

$$\lim_{x \rightarrow \infty} f(x) = 2 + \frac{x^2}{x^4 + 1} = 2$$
$$\lim_{x \rightarrow -\infty} f(x) = 2 + \frac{x^2}{x^4 + 1} = 2$$

So we have $y = 2$.

(d) Find the point(s) on f that are local extrema. But first verify that

$$f'(x) = -\frac{2x(x^4 - 1)}{(x^4 + 1)^2}.$$

Work: Yes, you should be able to verify this. If you can't, you need to review algebra. Now, to find extrema, we'll first need to find the critical points. Since $x^4 + 1$ is never zero, I just need to solve $2x(x^4 - 1) = 0$, which easily yields $\{-1, 0, 1\}$. Simple sign analysis of the first derivative, tells us that the function is increasing on $(-\infty, -1) \cup (0, 1)$ and is decreasing on $(-1, 0) \cup (1, \infty)$.

The minimum point is global and is (0, 2).

The maximum points (there are two) are global and are (-1, 2.5) and (1, 2.5).

(e) Find the interval(s) where f is increasing.

Work: From (4) above, simple sign analysis of the first derivative, tells us that the function is increasing on $(-\infty, -1) \cup (0, 1)$.

(f) Find the interval(s) where f is decreasing.

Work: From (4) above, simple sign analysis of the first derivative, tells us that the function is decreasing on $(-1, 0) \cup (1, \infty)$.

(g) Find the inflection point(s) on f . But first verify that

$$f''(x) = \frac{2(3x^8 - 12x^4 + 1)}{(x^4 + 1)^3}.$$

Work: Yes, you should be able to verify this. If you can't, you need to review algebra ... *AGAIN!* Now, to find points of inflection, we'll first need to find where the second derivative is zero or undefined. Since $x^4 + 1$ is never zero, I just need to solve $3x^8 - 12x^4 + 1 = 0$, which is a eighth-degree polynomial. I suggest that making a substitution, $u = x^4$, is helpful and is exactly what was presented in MTH119.

$$\begin{aligned} u &= x^4 \\ 3x^8 - 12x^4 + 1 &= 0 \\ 3u^2 - 12u + 1 &= 0 \\ u &= \frac{6 \pm \sqrt{33}}{3} \\ x &= \pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}} \end{aligned}$$

There's actually four inflection points and they are of the form:

$$\left(\pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}, f\left(\pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}\right) \right)$$

(h) Find the interval(s) where f is concave up.

Work: Okay, things are getting *nasty*. But there's point to be made, and that's exactly what I am trying to get across here. Now is probably the time to be using technology to help. So please consider learning whatever technology you have access to. For example, calculators are quite capable of solving equations, finding derivatives, and creating graphs. Furthermore, ECC provides access to Mathematica in the computer labs on the third floor.

One of the greatest movies ever made (*2001: A Space Odyssey*), in my humble opinion, had a religious sub-theme that actually made the point that our evolution as a species depends on the intelligent use of tools. Essentially those that can't adapt to a changing tool-set are doomed ... so how does one learn a tool? Some watch in

awe, but just take a look at the beginning of *2001: A Space Odyssey* to see a prime example that watching others use tools to their advantage is a very bad strategy. If anyone is interested, I have a DVD copy of *2001: A Space Odyssey* for anyone who wants to watch. As Clarke wrote in 1972: “Quite early in the game I went around saying, not very loudly, ‘MGM doesn’t know this yet, but they’re paying for the first \$10,000,000 religious movie.’ ”

Okay, finally the answer is:

$$\left(-\infty, -\sqrt[4]{\frac{6 + \sqrt{33}}{3}} \right) \cup \left(-\sqrt[4]{\frac{6 - \sqrt{33}}{3}}, \sqrt[4]{\frac{6 - \sqrt{33}}{3}} \right) \cup \left(\sqrt[4]{\frac{6 + \sqrt{33}}{3}}, \infty \right).$$

- (i) Find the interval(s) where f is concave down.

Work: Enough talk already! Here’s the answer:

$$\left(-\sqrt[4]{\frac{6 + \sqrt{33}}{3}}, -\sqrt[4]{\frac{6 - \sqrt{33}}{3}} \right) \cup \left(\sqrt[4]{\frac{6 - \sqrt{33}}{3}}, \sqrt[4]{\frac{6 + \sqrt{33}}{3}} \right).$$

3. Given the following graph.

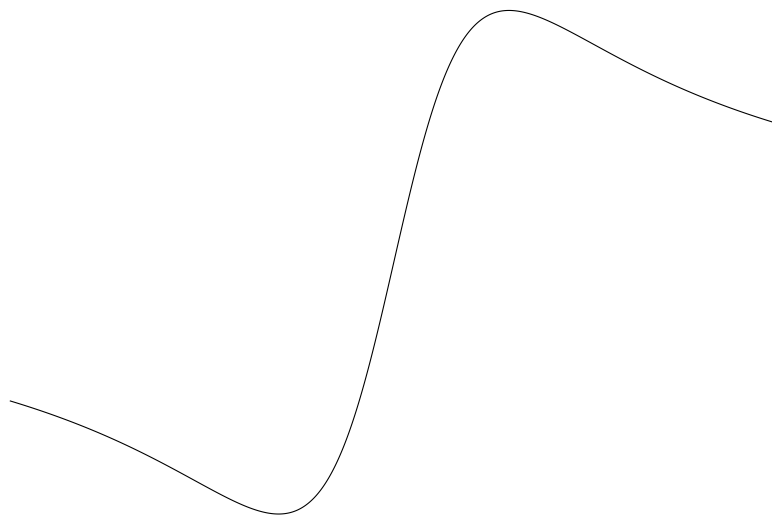


Figure 7: Partial graph of $f(x) = \frac{x}{x^2 + 9}$.

- (a) What is the domain?
- (b) Are there any x -intercepts?

- (c) Are there any y -intercepts?

- (d) Is there any symmetry?

- (e) Find and simplify $f'(x)$.

- (f) Find the point(s) on f that are local extrema.

- (g) Find the interval(s) where f is increasing.

- (h) Find the interval(s) where f is decreasing.

- (i) Find and simplify $f''(x)$.

(j) Find the inflection point(s) on f .

(k) Find the interval(s) where f is concave up.

(l) Find the interval(s) where f is concave down.

Answers follow.

(a) What is the domain?

Work: All real numbers, that is \mathbb{R} .

(b) Are there any x -intercepts?

Work: Yes , $f(x) = 0$ when $x = 0$, so the x -intercept is $(0, 0)$.

(c) Are there any y -intercepts?

Work: Yes , $f(0) = 0$, so the y -intercept is $(0, 0)$.

(d) Is there any symmetry?

Work: Yes , since $f(x) = -f(-x)$ the symmetry is odd .

(e) Find and simplify $f'(x)$.

Work:

$$\begin{aligned} f'(x) &= \frac{(x^2 + 9) - x(2x)}{(x^2 + 9)^2} \\ &= \frac{9 - x^2}{(x^2 + 9)^2} \\ &= \frac{(3 - x)(3 + x)}{(x^2 + 9)^2} \end{aligned}$$

(f) Find the point(s) on f that are local extrema.

Work: Simple sign analysis along with looking at the graph is all one needs to do. The $\boxed{\text{global maximum is } (3, 1/6)}$, and the $\boxed{\text{global minimum is } (-3, -1/6)}$.

(g) Find the interval(s) where f is increasing.

Work: From work on (6). $\boxed{(-3, 3)}$

(h) Find the interval(s) where f is decreasing.

Work: From work on (6). $\boxed{(-\infty, -3) \cup (3, \infty)}$

(i) Find and simplify $f''(x)$.

Work: It's getting tougher, but please make sure that your first derivative is correct or you're wasting your time.

$$\begin{aligned} f''(x) &= \frac{(x^2 + 9)^2 (-2x) - 2(x^2 + 9)(2x)(9 - x^2)}{(x^2 + 9)^4} \\ &= \frac{2x(x^2 + 9)[(x^2 + 9)(-1) - 2(9 - x^2)]}{(x^2 + 9)^4} \\ &= \boxed{\frac{2x(x^2 - 27)}{(x^2 + 9)^3}} \end{aligned}$$

(j) Find the inflection point(s) on f .

Work: Simple sign analysis along with looking at the graph is all one needs to do. The points of inflection are: $\boxed{(-3\sqrt{3}, -\sqrt{3}/12), (0, 0), \text{ and } (3\sqrt{3}, \sqrt{3}/12)}$.

(k) Find the interval(s) where f is concave up.

Work: From work on (10). $\boxed{(-3\sqrt{3}, 0) \cup (3\sqrt{3}, \infty)}$

(l) Find the interval(s) where f is concave down.

Work: From work on (10). $\boxed{(-\infty, -3\sqrt{3}) \cup (0, 3\sqrt{3})}$